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On the Problem of Hydrodynamic Stability. I. Uniform Shearing Motion in a Viscous Fluid

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VI. *On the Problem of Hydrodynamic Stability.*—I. *Uniform Shearing Motion in a Viscous Fluid.*

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Contents.

Introduction and Summary.

1. Except in a few very simple cases, the equations which govern the motion of a viscous fluid have so far defied analysis. Their difficulty comes mainly from the fact that they are not linear, so that the principle of superposition cannot be employed, as in many branches of mathematical physics, to construct solutions by the method of series or of singularities. For the same reason the flow pattern in the neighbourhood of a moving body must alter when the speed of the body is changed, and it follows that any exact determination of the pattern will be restricted to some definite speed.

As a matter of fact, no precise determination of this kind exists, except in cases where the motion is indefinitely slow. But the form of the equations gives no reason for doubting the possibility of "steady" motion (in which the velocities are functions only of position) in every case of flow past fixed and rigid boundaries. Now in experiment it is found (unless the velocities are very small) that eddying or periodic motions always occur. Thus the conclusion seems inevitable that a steady motion may become unstable as the rate of flow is increased, in the sense that accidental disturbances, if of suitable type, will persist.

2. The occurrence of eddies (or of "turbulent" motion) is intimately related with the important problem of the resistance of fluids, and it is therefore not surprising that the question of hydrodynamic stability has attracted much attention. Fairly obvious lines of investigation are suggested by the cognate theory of elastic stability. Starting from a known solution for steady motion, we imagine a disturbance to have been effected by some cause which it is not necessary to specify, and we examine, in the light of the

general equations, the history of the disturbance after it has become "free." The steady motion is judged to be stable if we can show that every disturbance, whatever its type, tends ultimately to vanish, and neutral or unstable if any type is found to persist or to increase. Since the general equations of motion are not linear, the question of stability may depend upon the *scale* of the disturbance; but if the disturbance is assumed to be infinitesimal (as is customary in elastic problems), we may omit infinitesimals of the second order from the equations which govern the disturbed motion, and then the history of the disturbance (within the assumption stated) is independent of its magnitude.

3. A knowledge of the initial (steady) motion is of course presumed, and this requirement limits the scope of our examination to those few and simple cases in which the equations of steady motion have been solved exactly. The following list is believed to cover all solutions which are known* :—

(A) The laminar flow of a fluid, under uniform pressure gradient or body forces, between two fixed parallel planes (Lamb, 'Hydrodynamics,' § 330).

(B) The laminar flow of a fluid between two plane and parallel boundaries which have a uniform velocity relative to one another in a direction parallel to their planes (*ibid.*, § 330, *a*).

(C) The rectilinear flow of a fluid, under uniform pressure or body forces, through a straight pipe of uniform section (*ibid.*, §§ 331, 332).

(D) Two-dimensional rotatory motion of a fluid about a fixed axis, between two concentric cylinders of infinite length (*ibid.*, § 333).

The question of stability in case (D) has been examined by G. I. Taylor,† who determined not only the conditions under which the two-dimensional motion will become unstable but also the nature of the motion which then sets in, and has completely substantiated his conclusions by experiment. Before the publication of these results the other three problems had attracted more attention,—presumably on account of their greater analytical simplicity, because a satisfactory verification of theory by experiment would here, in the nature of the case, present serious difficulties. In regard to the simplest motion (B) there appears to be general agreement with the conclusion of Kelvin,‡ that the laminar flow is in all cases stable for infinitely small disturbances,

* It will be appreciated that approximate solutions, or solutions limited to very slow motions, are useless for an examination of stability.

† 'Phil. Trans.,' A, vol. 223 (1923), pp. 289–343.

‡ 'Phil. Mag.,' vol. 24 (1887), pp. 188–196 and 272–278; 'Collected Papers,' vol. IV, No. 34 (1887), p. 321. Kelvin quotes, in support of his conclusion, the following descriptions of observed results by Osborne Reynolds ('Phil. Trans.,' vol. 174 (1883), pp. 955–6).

"The fact that the steady motion breaks down suddenly shows that the fluid is in a state of instability for disturbances of the magnitude which cause it to break down. But the fact that in some conditions but that for disturbances exceeding a certain limit of size the motion becomes unstable, these limits of stability being narrower the smaller the viscosity. On the other hand, this conclusion has not been held to be satisfactorily established, even in regard to two-dimensional disturbances; RAYLEIGH has described the problem as "of no ordinary difficulty,"* and LAMB remarks (1924) that "Most writers who have attacked the subject are disposed to regard the conclusion as probable, though as yet hardly demonstrated."†

4. This paper describes work which has been done in an attempt to examine the stability of the motion (B). It should be explained at the outset that we have concentrated attention on this particular problem solely because our interest is centred for the present in methods rather than in results. Our aim has been to develop a general theory, of the kind which RAYLEIGH employed so successfully in relation to problems of vibration, *whereby the critical velocity of steady flow might be estimated approximately in cases where exact solutions are not obtainable*; and in order to test the practical accuracy of such a theory when found, it will evidently be necessary to compare its results with those of more exact analysis. Now although the motion (B) is not of any particular interest in itself, it offers what would appear, *a priori*, to be the simplest case for theoretical investigation, because the formal solution is expressible in terms of known functions. Thus it is for the moment the obvious test case.‡

We have not yet succeeded in our attempt to answer the question, whether the steady motion is in fact stable or unstable. This paper is presented (1) as a critical review of previous work, indicating reasons why we regard the question as still awaiting a really satisfactory answer, and (2) as an interim report giving certain preliminary results which seem to have an interest of their own.

5. The scheme of the paper is as follows: Attention is confined to disturbances which are two-dimensional, so that the quantities involved do not vary with z, when the direction Oz is parallel to the plane boundaries and perpendicular to the direction of their relative motion. In most parts of the paper, but not everywhere, the disturbance is further restricted to be infinitesimal.

We begin by considering previous work on the problem. Section I deals with KELVIN's original investigations, and the criticisms to which they have been subjected by

it will break down for a large disturbance, while it is stable for a smaller disturbance, shows that there is a certain residual stability so long as the disturbances do not exceed a given amount."

"And it was a matter of surprise to me to see the sudden force with which the eddies sprang into existence, showing a highly unstable condition to have existed at the time the steady motion broke down."

"This at once suggested the idea that the condition might be one of instability for disturbance of a certain magnitude and stable for smaller disturbances."

It should be remarked that REYNOLDS' experiments were made with pipes of circular section.

* 'Collected Papers,' vol. 6, p. 275 (1914).

† 'Hydrodynamics,' 5th ed., § 368.

‡ Most of previous work on hydrodynamic stability has been concerned with this problem.

RAYLEIGH and by ORR; with a somewhat similar investigation by OSEEN; and with the well-known method of OSBORNE REYNOLDS. It is concluded that these researches have not afforded a complete answer to the question of stability, and that the same is true of REYNOLDS' method as modified by ORR.

Section II examines the possibility of inferring stability from an integral equation differing in form from that which is used in REYNOLDS' method. At one time we thought that our problem could be solved on these lines, but reasons are now given why this belief must be discarded. The only conclusion which emerges from the discussion is that the stability problem for a viscous fluid cannot be simplified by partial neglect of the viscosity, after the manner of some of RAYLEIGH's earlier investigations.

6. Section III describes the "Method of Normal Co-ordinates," which entails a study of disturbances varying according to a simple exponential factor of the time. It is shown that, on assumptions which are usually regarded as permissible from the physical standpoint, the method may be used to obtain a complete solution (for infinitesimal disturbances) in the special case of the problem where the plane boundaries are at rest. The method is then applied to the general case (with moving boundaries). It is found that certain properties (the so-called "conjugate relations") which are ordinarily possessed by "normal" solutions are here no longer satisfied, with the result that we can no longer show that the time factors are always negative, or even that they are necessarily real. Again, it is found that the method of expansion employed in the special case, to express an arbitrary disturbance as a series of disturbances of "normal" type, also ceases to apply. This last result suggests the question, whether stability would in fact be demonstrated if we could prove that all "normal" disturbances (*i.e.*, solutions characterised by simple exponential time factors) inevitably decay. We state our reasons for believing that the proof would be sufficient as regards stability for *infinitesimal* disturbances. This appears to be the view of most workers on the subject.

7. The only paper we have found which investigates in detail the dependence of the time factor upon "REYNOLDS' number" is that of L. HOPF ('Ann. der. Phys.,' 1914). HOPF's investigation indicates stability for infinitesimal disturbances; but it is subject to some quantitative uncertainty, estimated by him as never greater than 10 per cent., arising from the fact that he replaces Bessel functions by their semi-convergent series and retains only the first terms of these. We believe that it reveals with sufficient accuracy all the main features of the problem, and that HOPF's conclusion in regard to the stability is correct. But the failure of the conjugate relations between normal disturbances (§ 6) makes it necessary to regard the use of approximations with considerable distrust. RAYLEIGH, commenting on the work of HOPF and VON MISES, remarked: "Doubtless the reasoning employed was sufficient for the writers themselves, but the statements of it put forward hardly carry conviction to the mere reader."*

* 'Collected Papers,' vol. 6, p. 275 (1914).

8. It thus appeared that a more exact investigation of the "normal" disturbances was called for. A grant made to one of us by the kindness of the Director of Scientific Research, Air Ministry, enabled us to collaborate in this work, which forms the matter of Section IV. The equation governing normal disturbances can be solved in series when the index of the exponential time factor is assumed; and if the index has been given a possible value, solutions obtained in this way can be combined so as to satisfy the boundary conditions. Hence, by a process of trial and error, values of the time factor can be determined. The method is extremely laborious, and it becomes impracticable when "REYNOLDS' number" exceeds a fairly small value; but within its limitations it yields results which are reasonably definite and certainly interesting.

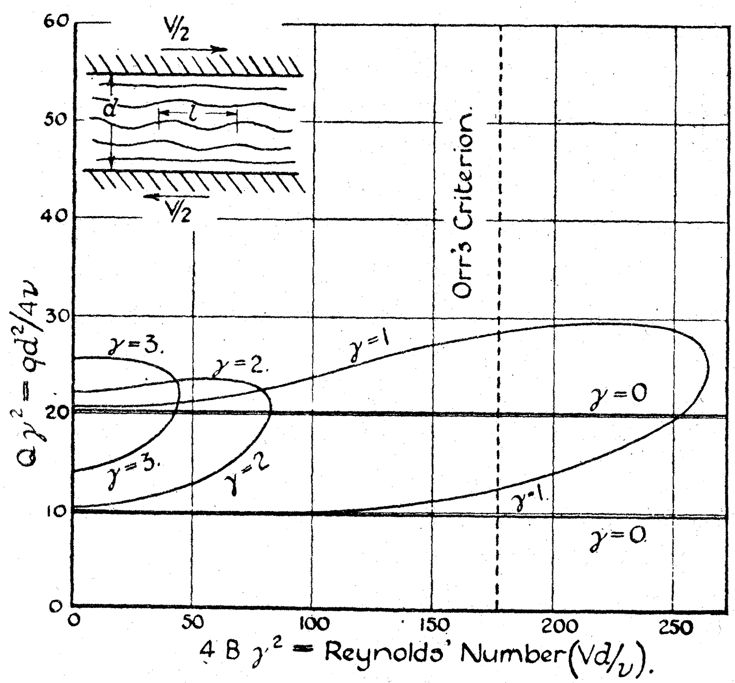

FIG. 1.

The general nature of these results is exhibited by fig. 1. In that diagram the slowest rate at which a normal disturbance can decay is related with the "REYNOLDS' number" of the steady motion by a series of curves. As REYNOLDS' number we have taken the quantity Vd/ν, where V is the relative velocity of the two plane boundaries, d their distance apart, and ν the kinematic viscosity of the fluid. Ordinates in the diagram represent the non-dimensional quantity $q\,d^2/4\nu$, where e^{-qt} is the time factor of the normal disturbance; thus a positive value of q implies that the corresponding normal disturbance tends ultimately to vanish. The disturbance is assumed to be simple harmonic as regards its variation with distance measured parallel to the plane boundaries; representing the wave-length in this direction by l, we have denoted by γ the non-dimensional quantity $\pi d/l$, which may evidently have any value. The

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rate of decay depends upon γ, and curves have accordingly been drawn, in fig. 1, for the cases $\gamma = 0, 1, 2, 3$. The small sketch in the left-hand top corner of the diagram will serve to explain the foregoing symbols.

9. The curves in fig. 1 are entirely confined to positive values of q, indicating subsidence of all normal disturbances within the range of the investigation; but this result throws little new light on the question of stability, since it was known already that all normal disturbances must decay when REYNOLDS' number has a value less than 177 ("ORR's criterion," indicated in the diagram by a dotted line). The most important feature of the diagram is the looped form of the curves, which has no counterpart in ordinary problems of vibration theory, and to which attention does not seem to have been drawn before. On its discovery, it at once became a matter of interest to investigate the way in which the type of the disturbance changes as we pass round any given loop; this matter is fully discussed in Section IV.

10. The closure of the loops indicates that for larger values of REYNOLDS' number the time factors will assume complex values: whether their real parts will be positive or negative is a question which we do not attempt to investigate in this paper. Physically, a complex time factor means that the normal disturbance (regarded as superposed upon the steady laminar motion) is no longer stationary in position,* but travels with some finite velocity along the channel; it can be shown that this velocity will be intermediate between the velocities of the plane boundaries,—that is, it will be equal to the velocity of the steady stream at some point within the fluid field.† On the mathematical side it means that there is a new variable to be introduced in deriving solutions by a process of trial and error; and the method employed in this paper, which has become very laborious within the range of our investigation, accordingly becomes definitely impracticable as a line of further advance.

11. In a subsequent paper we shall describe methods by which this deadlock has been partially overcome; the present paper is to be regarded as merely preliminary, indicating the complexity of the problem and disposing of certain methods which have been proposed for its solution. Our main conclusions may be summarised as follows :—

(*a*) None of the methods which have been suggested as alternatives to the method of normal co-ordinates seem to be applicable to the investigation of our problem.

(*b*) The method of normal co-ordinates appears to be satisfactory as regards the investigation of stability for infinitesimal disturbances, in the sense that a sufficient proof of stability will have been provided if we can show that all disturbances of normal type have a decreasing time factor. Our discussion of this question makes no claim to mathematical rigour; but we see no reason why the method should be less trustworthy as applied to our problem than in others to which it has been applied without objection.

* *i.e.*, when the plane boundaries move with equal and opposite velocities.

† This result is due to ORR. *Cf.* Section IV, § 4.

(*c*) Within the range of REYNOLDS' numbers which is covered by our investigation, all normal disturbances have a decreasing time factor.

(*d*) The time factor varies with REYNOLDS' number in a manner which is extremely complicated. The looped diagrams have no counterpart in ordinary problems of vibration theory.

It remains to acknowledge our indebtedness to the Director of Scientific Research, Air Ministry, for the grant to which reference has been made in §8, and to Professors H. Lamb and G. I. Taylor for their interest in the work, and for valuable criticism and advice.

BIBLIOGRAPHY.

1. HELMHOLTZ, 'Phil. Mag.,' vol. 36 (1868), p. 337.
2. HOPF, 'Annalen der Physik,' vol. 44 (1914), p. 1.
3. KELVIN, 'Phil. Mag.,' vol. 24 (1887), pp. 188–196.
4. KELVIN, 'Phil. Mag.,' vol. 24 (1887), pp. 272–278.
5. KORTEWEG, 'Phil. Mag.,' vol. 16 (1883), p. 112.
6. LAMB, 'Hydrodynamics' (5th edition).
7. LAMB, 'Aeronautical Research Committee,' R. and M. 1084 (1926).
8. LORENTZ, 'Abhandlungen über theoretische Physik,' vol. 1 (1907), p. 43.
9. v. MISES, 'Jahresb. d. deutschen Math. Ver.,' vol. 21 (1912), p. 241.
10. ORR, 'Proc. Roy. Irish Acad.,' vol. 27 (1907), pp. 9–138.
11. OSEEN, 'Arkiv für Math. Ast. och Fysik, Upsala,' vol. 7, No. 15 (1911).
12. REYNOLDS, 'Phil. Trans.,' vol. 174 (1883), p. 935.
13. REYNOLDS, 'Phil. Trans.,' A, vol. 186 (1895), p. 123.
14. RAYLEIGH, 'Collected Papers,' vol. I. No. 58 (1879).
15. RAYLEIGH, 'Collected Papers,' vol. I, No. 66 (1880).
16. RAYLEIGH, 'Collected Papers,' vol. III, No. 144 (1887).
17. RAYLEIGH, 'Collected Papers,' vol. III, No. 194 (1892).
18. RAYLEIGH, 'Collected Papers,' vol. IV, No. 216 (1895).
19. RAYLEIGH, 'Collected Papers,' vol. VI, No. 377 (1913).
20. RAYLEIGH, 'Collected Papers,' vol. VI, No. 388 (1914).
21. SOMMERFELD, 'Atti del Cong. Intern. dei Math.' (Roma, 1909), vol. III, p. 116.
22. SOUTHWELL, 'Phil. Mag.,' vol. 48 (1924), p. 540.
23. STOKES, 'Camb. Phil. Soc. Trans.,' vol. 8 (1843), p. 105.
24. TAYLOR, 'Phil. Trans.,' A, vol. 223 (1923), p. 289.

SECTION I.—*Review of earlier Investigations. Methods of* KELVIN, OSEEN *and* OSBORNE REYNOLDS.

Formulation of the Problem.

1. This paper is concerned with the stability (or instability) of the steady motion which has been described in the introductory section as "Motion (B)". A viscous incompressible fluid occupies the space between two plane and parallel boundaries of infinite extent. These boundaries move with uniform velocities in a direction parallel to their planes, and each impresses its own velocity upon the fluid with which it is in contact; the relative velocity of the two boundaries is V. When V is small, the fluid has a velocity which is everywhere parallel to V; the motion is "steady,"—that is to say, it does not vary with the time. Our problem is to decide whether, when V has a large value, this steady regime is liable to give place to "turbulent" motion, in which the velocity of the fluid at any point fluctuates in direction between one instant and another. Dimensional theory indicates that turbulent motion, if it is ever liable to occur, will do so when the non-dimensional quantity $\mathrm{V}d/\nu$, termed "REYNOLDS' number" for the steady motion considered, attains a certain value. d denotes the distance between the plane boundaries, and ν the "kinematic viscosity" of the fluid; V, d and ν may be measured in any self-consistent system of units.

2. We shall now state the problem in mathematical form. Let Ox, Oy, Oz be a system of perpendicular axes, Oy being perpendicular to the plane boundaries and Ox along the direction of their relative motion. In the absence of body forces, the motion of the fluid will be governed by

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and two similar equations, together with

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the "equation of continuity," which expresses the condition that the fluid is incompressible. In (1) and (2), u, v, w are the component velocities in the directions x, y, z respectively, p is the "mean pressure" of the fluid, and ρ and ν are the density and kinematic viscosity. At the boundaries we must have

$$u=\mathrm{U}, \qquad v=0, \qquad w=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where U is the velocity of the boundary in question.

It is clear that all the conditions will be satisfied if we take (3) to hold at every point in the fluid, and write

$$\mathrm{U}=\alpha+\beta y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where α and β are constants adjusted to give the correct boundary values for U. That is to say, a "steady" solution, represented by (3) and (4), exists for *all* values of the relative velocity V. We notice that $V = \beta d$, so that REYNOLDS' number (§ 1) can be expressed in the form

$$R = \beta d^2/\nu. \quad \quad (5)$$

Now let us imagine that the steady regime is disturbed. Then we have initially ($t = 0$) velocities u_0, v_0, w_0 which must satisfy (2) at every point and (3) at the boundaries, but which will not satisfy (3) at other points in the fluid. Writing

$$u = U + u', \qquad v = v', \qquad w = w', \quad \quad (6)$$

where U is given by (4), we may say that u', v', w' define the "disturbance." Their initial values u_0', v_0', w_0' are given; at all times they satisfy an equation of the form (2) and the boundary conditions

$$u' = v' = w' = 0\,; \quad \quad (7)$$

after the disturbance has become "free"* they are governed by equations which may be obtained by substitution from (6) in the three equations of type (1).

The question of stability turns on the behaviour of the disturbances as controlled by these conditions. If any solution for u', v', w' can be found which does not tend to zero as $t \to \infty$, the steady motion is said to be unstable, in the sense that a disturbance of this type will persist: if we can show that u', v', w' inevitably come to zero, we may judge the steady motion to be stable.

3. In this paper we shall confine attention to disturbances which are two-dimensional, so that all quantities in (1) are independent of z, and $w = 0$. The last of the equations (1) is then satisfied identically, and the other two become

$$\left.\begin{aligned} \frac{\partial u'}{\partial t} + (U + u')\frac{\partial u'}{\partial x} + v'\frac{\partial}{\partial y}(U + u') &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u', \\ \frac{\partial v'}{\partial t} + (U + u')\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y} \qquad\qquad &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v', \end{aligned}\right\} \quad (8)$$

∇^2 now denoting the operator $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$. The equation of continuity (2) takes the form

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0, \quad \quad (9)$$

* We are not concerned with the mechanism by which the disturbance is set up, except to postulate that after the instant ($t = 0$) it ceases to be operative. Whilst it is acting ($t < 0$), the disturbance may be said to be "forced"; afterwards ($t > 0$) it is said to be "free."

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and so permits the introduction of a "stream function" ψ, defined by

$$u' = -\frac{\partial \psi}{\partial y}, \qquad v' = \frac{\partial \psi}{\partial x}. \quad \ldots \ldots \quad (10)$$

The relations (7) reduce to two, which may be written in the form

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0, \quad \ldots \ldots \quad (11)$$

at the plane boundaries.

Eliminating p from (8) by cross-differentiation with the aid of (9), we obtain the equation

$$\frac{\partial \zeta}{\partial t} - \nu \nabla^2 \zeta + (\mathrm{U} + u') \frac{\partial \zeta}{\partial x} + v' \frac{\partial \zeta}{\partial y} = 0, \quad \ldots \ldots \quad (12)$$

where

$$\zeta = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = \nabla^2 \psi, \text{ by (10)}. \quad \ldots \ldots \quad (13)$$

When the disturbance is infinitesimal, so that quantities of the second order in ψ may be neglected, equation (12) reduces to

$$\frac{\partial \zeta}{\partial t} - \nu \nabla^2 \zeta + \mathrm{U} \frac{\partial \zeta}{\partial x} = 0. \quad \ldots \ldots \quad (14)$$

4. In an earlier paper dealing with our present problem,* attention was called to the necessity of imposing further conditions supplementary to (11). "In seeking to determine the conditions (if any exist) under which an accidental disturbance of the steady motion will tend to persist or to increase, we must ensure that such persistence is not in reality an effect of suitably varying pressure differences, implied in our solution, at the *ends* of the fluid field. This will be done if we . . . . . take the length of the fluid field to be infinite and the disturbance to be simply harmonic in x: our problem then becomes a limiting case of the problem of shearing motion between two concentric cylinders, in which the radii of the cylinders are infinite."†

That paper was concerned only with infinitesimal disturbances. In the general case of our problem we have to consider finite disturbances which (on account of the non-linear form of the governing equations) cannot be analysed into independent disturbances of different wave-lengths. So we can no longer assume that the disturbance is simple harmonic in x; but on the other hand we must impose some condition of an equivalent nature.

We may regard the plane boundaries of our problem as the limiting forms of two concentric circular tubes, each of which is bent in a circle and joined on itself so as to

* 'Phil. Mag.,' vol. 48, p. 540 (1924).

† *Loc. cit.*, p. 544.

form a hollow tore or anchor ring. Fig. 2 illustrates the system here contemplated. The two concentric tubes are shown in section, and the annular space between them (shown shaded) is imagined to be filled by the fluid, which in the steady motion has a velocity purely normal to the section,—*i.e.*, *along* the tubes, in the direction denoted by x. Indefinite extension of the fluid field in the direction of z can be reproduced by making infinite the mean radius of the *tubes* (r in fig. 2); and indefinite extension in

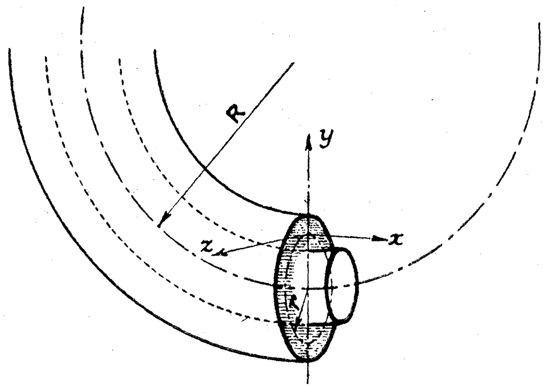

Fig. 2.

the direction of x can be reproduced, similarly, by making infinite the mean radius of the *tores* (R in fig. 2).

It is now unnecessary to make any assumption in regard to the nature of the disturbance, provided that we postulate that in any integration extending throughout the whole of the fluid field we may write

$$\int_{-\infty}^{\infty} \frac{\partial \phi}{\partial x}\, dx = 0, \quad \int_{-\infty}^{\infty} \frac{\partial \phi}{\partial z}\, dz = 0, \quad \ldots \ldots \ldots \quad (15)$$

where ϕ is any function of the fluid velocities, or pressures, which is necessarily single-valued in the multiply-connected fluid field of the system shown in fig. 2. When (as in the present paper) attention is confined to two-dimensional disturbances, the second of these conditions is satisfied already.

This device for giving precision to the boundary conditions of our problem was (in effect) employed by Rayleigh in a similar connection.*

The Steady Motion is Stable for Laminar Disturbances.

5. We may remark at this point that when the disturbance is laminar ($v' = w' = 0$) stability can be inferred at once. For the second and third of the equations of motion (1) will now require p to be independent both of y and of z, and (2) shows that u'

* 'Collected Papers,' vol. 3, pp. 578–581 (1892).

is then independent of x. It follows from the first of equations (1) that $\partial p/\partial x$ must be independent of x, and then the first of the conditions (15) which have just been imposed shows that $\partial p/\partial x$ is everywhere zero. Accordingly we have from (1) the relation

$$\frac{\partial u'}{\partial t} = \nu \left(\frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Again, u' must vanish at either boundary. Hence, multiplying (16) by u' and integrating throughout the fluid field, we obtain the equation

$$\frac{\partial}{\partial t}\iint u'^2\, dy\, dz = -2\nu \iint \left\{\left(\frac{\partial u'}{\partial y}\right)^2 + \left(\frac{\partial u'}{\partial z}\right)^2\right\} dy\, dz,$$

in virtue of the second of the conditions (15). This result shows that u' comes ultimately to zero at every point in the field.

History of the Problem.

6. The earliest discussion of hydrodynamic stability appears to be that given by HELMHOLTZ, who showed that, in an inviscid liquid, a surface at which the velocity is discontinuous will be essentially unstable.* In 1879, RAYLEIGH applied a method due to KELVIN to investigate more precisely the character of the instability.† It soon appeared that the calculations failed in one important respect to correspond with the facts, and the explanation was suspected to lie in the assumption of discontinuous changes of velocity, which in a real fluid, by reason of viscosity, must instantly disappear.‡ Accordingly, in succeeding papers,§ RAYLEIGH modified his assumptions in regard to the steady motion, and dealt with laminar systems in which, although the vorticity varied abruptly, the velocities were taken to be continuous. His most general conclusion relates to infinitesimal disturbances in two dimensions : " The steady motion of a non-viscous liquid in two dimensions between fixed parallel plane walls is stable provided that the velocity U, everywhere parallel to the walls and a function of y only, is such that d^2U/dy^2 is of one sign throughout, y being the co-ordinate measured perpendicularly to the walls."‖

* 'Phil. Mag.,' vol. 36, p. 337 (1868). The possible existence of unstable solutions of the equations of motion seems to have been first suggested by STOKES, 'Camb. Phil. Soc. Trans.,' vol. 8, p. 105 (1843).

† 'Collected Papers,' vol. 1, No. 58 (1879).

‡ RAYLEIGH, 'Collected Papers,' vol. 1, No. 66, p. 475 (1880).

§ 'Collected Papers,' vol. 1, No. 66 (1880) ; vol. 3, No. 144 (1887) and No. 194 (1892) ; vol. 4, No. 216 (1895). The whole investigation, and criticisms advanced by KELVIN and LOVE, have been fully reviewed by ORR, 'Proc. R. Irish Acad.,' vol. 27, pp. 9–138 (1907).

‖ 'Collected Papers,' vol. 6, p. 266 (1914)

According to this conclusion, the steady motion now under consideration would be judged to be either stable or "neutrally stable,"—certainly not unstable. This result seems not to have been expected by RAYLEIGH,* and he advanced several suggestions to explain the supposed discrepancy between theory and experiment. The most important of these may be quoted: " . . . the impression upon my mind is that the motions calculated . . . for an absolutely inviscid liquid may be found inapplicable to a viscous liquid of vanishing viscosity, and that a more complete treatment might even yet indicate instability, perhaps of a local character, in the immediate neighbourhood of the walls, when the viscosity is very small."†

In 1883, OSBORNE REYNOLDS published a remarkable account of experimental observations relating to the appearance of turbulence in fluid flowing down a long straight pipe.‡ His results attracted great interest, notably on the part of RAYLEIGH and KELVIN. At RAYLEIGH's suggestion "the Criterion of the Stability and Instability of the Motion of a Viscous Fluid" was proposed as the subject for an Adams prize essay; shortly afterwards the subject was taken up by KELVIN, whose investigations will now be described. KELVIN's conclusion was that the steady motion is wholly stable for infinitesimal disturbances, whatever may be the value of the viscosity; but that when the disturbances are finite the limits of stability become narrower and narrower as the viscosity diminishes.

KELVIN's *Investigations.*

7. KELVIN employed two methods: the first§ a special method applicable only to the problem of this paper, the second|| more general and also applicable to other problems.

In a paper published in 1892, RAYLEIGH¶ indicated objections to KELVIN's second (general) method: " . . . If I rightly understand it, the process consists in an investigation of forced vibrations of arbitrary (real) frequency, and the conclusion depends upon a tacit assumption that if these forced vibrations can be expressed in a periodic form, the steady motion from which they are deviations cannot be unstable. A very simple case suffices to prove that such a principle cannot be admitted." RAYLEIGH's objection has been supported by ORR, and seems to have been accepted by KELVIN**; accordingly it need not be further considered here.

KELVIN's first (special) method appears to have been accepted by RAYLEIGH

* 'Collected Papers,' vol. 3, p. 576 (1892).

† *Ibid.*, p. 582.

‡ 'Phil. Trans.,' vol. 174, p. 935 (1883). A quotation from this paper has been made in the Introductory Section, § 3.

§ 'Phil. Mag.,' vol. 24, pp. 188–196 (1887).

|| 'Phil. Mag.,' vol. 24, pp. 272–278 (1887).

¶ 'Collected Papers,' vol. 3, p. 582 (1892).

** RAYLEIGH, 'Collected Papers,' vol. 6, p. 267 (1914).

originally* ; but later criticisms, advanced by ORR, showed that it too must be rejected as a decisive proof of stability. In this method the disturbance is restricted to be infinitesimal, but it need not be two-dimensional. We shall, however, explain it here in relation to disturbances of two-dimensional type, which are governed by equation (14) ; accordingly we shall not follow KELVIN's treatment in detail.

If in equation (14) we were to make ν zero, then since U is a function of y only the resulting equation could be solved in the form

$$\zeta = \mathrm{F}\,\{(x - \mathrm{U}t),\, y\},$$

where the form of F is defined by the initial conditions (for $t = 0$). In the absence of viscosity ($\nu = 0$) the fluid can " slip " at the boundary ; hence there is only one boundary condition to be satisfied, and from ζ, as given here, a solution for ψ can be derived.

On the basis of this solution KELVIN assumed for trial in equation (14) the expression

$$\zeta = \mathrm{T}e^{i\{\lambda y + k(x - \mathrm{U}t)\}}, \qquad (17)$$

where T is a function of t only. On substitution in (14) it is found that T must satisfy the equation

$$\frac{d\mathrm{T}}{dt} + \nu\,\{k^2 + (\lambda - k\beta t)^2\}\,\mathrm{T} = 0,$$

in which β has been written for $d\mathrm{U}/dy$. Hence

$$-\log \mathrm{T} = \nu\,\{(k^2 + \lambda^2)\,t - k\lambda\beta t^2 + \tfrac{1}{3}k^2\beta^2 t^3 + \text{const.}\},$$

and the type solution (17) takes the form

$$\nabla^2\psi = \zeta = \mathrm{A}e^{-\nu t\,(k^2 + \lambda^2 - k\lambda\beta t + \frac{1}{3}k^2\beta^2 t^2) + i\,\{\lambda y + k\,(x - \mathrm{U}t)\}}, \qquad (18)$$

which becomes, when $t = 0$,

$$\zeta = \mathrm{A}\,e^{i\,(kx + \lambda y)}.$$

From (18) we derive, as the corresponding expression for ψ,

$$\psi = -\frac{\mathrm{A}e^{-\nu t\,(k^2 + \lambda^2 - k\lambda\beta t + \frac{1}{3}k^2\beta^2 t^2) + i\,\{\lambda y + k\,(x - \mathrm{U}t)\}}}{k^2 + (\lambda - k\beta t)^2}. \qquad (19)$$

This gives, when $t = 0$,

$$\psi = -\frac{\mathrm{A}}{k^2 + \lambda^2}\,e^{i\,(kx + \lambda y)}. \qquad (20)$$

KELVIN argued that any arbitrary initial disturbance can be analysed as a series of particular solutions of the type (20), values of λ being chosen which permit the satisfaction of the boundary conditions (now two in number, since ν does not vanish) when $t = 0$. The subsequent motion, for an unlimited field, would be given by a

* 'Collected Papers,' vol. 4, p. 209 (1895).

corresponding series of the functions (19); but since the latter functions do not, as t increases, continue to satisfy the boundary conditions, it is necessary to add a supplementary solution of (14), such that the combined solution satisfies the boundary conditions throughout all time. This supplementary solution (termed by him a "forced" disturbance) KELVIN proposed to construct "after the manner of Fourier" from a series of solutions which are simple exponential functions of the time. He assumed, for his type solution of this last class, that

$$\psi \propto e^{i(kx+\omega t)},$$

which makes ζ a function of the same form. Its governing equation is derived from (14) by substituting ik for $\partial/\partial x$ and $i\omega$ for $\partial/\partial t$.

KELVIN made no attempt to carry out in detail the analysis which would enable him, according to this method, to investigate the history of any representative initial disturbance. He argued that the supplementary solution is required to neutralise at the boundaries the values of ψ and $\partial\psi/\partial y$ which are given by the primary solution (19); and that, since the latter inevitably comes to zero when t is very large (the time factor is ultimately governed by the term $e^{-\frac{1}{3}\nu k^2\beta^2 t^3}$), the supplementary solution must also come to zero. Hence he concluded that the steady motion is stable.

8. Commenting on this argument, ORR remarks*: " . . . . it must, I think, be held that neither does it afford a proof of the stability of the motion." ORR argues that the supplementary solution is completely determined by the necessity of neutralising at the boundaries the values of ψ and $\partial\psi/\partial y$ as given by (19); and that KELVIN has assumed without justification that it consequently starts *from zero* at $t = 0$. If the supplementary solution is in fact to start from zero, solutions of the type (19) must not only satisfy both boundary conditions initially, but also be capable of representing any arbitrary initial disturbance.

Later ORR objects to KELVIN's assumption, that the supplementary solution will come asymptotically to zero as t increases to infinity *because this is true of the primary solution* (19). He says† that this can be asserted only of the *boundary values* of the supplementary solution; and he concludes that "Lord KELVIN has not proved stability, even for infinitesimal disturbances."‡ He gives examples to show that KELVIN's assumptions will be justified only if it is known that the fundamental free disturbances are stable,—and this, of course, is the question at issue.

It would seem that a slight modification of KELVIN's analysis enables us to meet ORR's first objection, that KELVIN's supplementary solution has not been proved to start from zero (everywhere) at $t = 0$. It appears (Section III) that any arbitrary disturbance (of infinitesimal magnitude) can be expressed in a series of component disturbances, each of which satisfies an equation of the type

$$[q + \nu\nabla^2]\nabla^2\psi = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

* 'Proc. R. Irish Acad.,' vol. 27, p. 85 (1907). † *Ibid.*, p. 88. ‡ *Ibid.*, p. 71.

together with the boundary conditions (11). Now by a slight extension of KELVIN's analysis it is possible to derive as a solution of (14) the expression

$$\zeta = \mathrm{A}\, e^{-\nu t\,(k^2+\lambda^2+\frac{1}{3}k^2\beta^2t^2)}\,[e^{\nu\beta k\lambda t^2}\cos\{k\,(x-\mathrm{U}t)+\lambda y\}+e^{-\nu\beta k\lambda t^2}\cos\{k\,(x-\mathrm{U}t)-\lambda y\}], \qquad (22)$$

where A is an arbitrary constant. Initially ($t = 0$) this reduces to the form

$$\zeta_0 = 2\mathrm{A}\cos kx\cos\lambda y,$$

and therefore satisfies (21) when λ is given an appropriate value. Similar expressions can be found which for ($t = 0$) reduce to the forms

$$\zeta_0 \propto \sin kx\cos\lambda y,$$

$$\zeta_0 \propto \sin kx\sin\lambda y$$

and

$$\zeta_0 \propto \cos kx\sin\lambda y\,;$$

and it follows that any arbitrary disturbance can be expressed in a series of terms of the type (22), *which series initially satisfies both boundary conditions of our problem.*

But the necessity for a supplementary ("forced") solution remains, because it is not possible to derive from (22) an expression for ψ which continues to satisfy *both* boundary conditions when t becomes finite. Hence, if one or more of the free disturbances (satisfying both boundary conditions) should be characterised by an exponentially increasing time factor,—and this is the question at issue,—it appears that the forced disturbance, although it eventually comes to zero at the boundaries, may increase indefinitely in other parts of the field.

OSEEN'S *Investigation.*

9. It thus appears that KELVIN'S researches must be regarded as having failed to demonstrate stability even for infinitesimal disturbances, although they make this conclusion probable. Since the difficulty, according to the argument just given, arises solely from the fact that (22) is not compatible with the satisfaction of both boundary conditions throughout all time, it evidently lies in the unknown action of the boundaries, from which vorticity must be "conducted" into the fluid (this is the physical explanation of the necessity of a supplementary solution) in order that the boundary conditions may be satisfied. KELVIN'S investigation fails because it omits to study the details of this boundary conduction.

The same objection applies to an investigation by OSEEN, in which a solution of (14) is obtained in the form*

$$\zeta\,(x,y,t) = \frac{\mathrm{C}\,e^{-\left[\frac{\{\xi-x+\frac{1}{2}\beta t\,(\eta+y)\}^2}{4\nu t\,(1+\frac{1}{12}\beta^2t^2)}+\frac{(\eta-y)^2}{4\nu t}\right]}}{4\pi\nu t\,(1+\frac{1}{12}\beta^2t^2)^{\frac{1}{2}}},$$

* OSEEN, 'Arkiv Math. Ast. Fys. Upsala,' vol. 7, No. 15 (1911). It is easily verified that the foregoing expression satisfies (14) when $\mathrm{U} = \beta y$, and that it reduces for ($t = 0$) to a vortex concentrated at one point (ξ, η).

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where

$$C = \iint \zeta_{(\xi, \eta, 0)}\, d\xi\, d\eta.$$

OSEEN makes no attempt to satisfy the conditions at the plane boundaries, stating that he has considered "the specially unfavourable case" in which the distance between the boundaries is infinitely great. But this surely means that he has neglected the only agency which can be conceived to produce instability,—by neutralising the dissipation which is inevitably caused by viscosity. RAYLEIGH, commenting on OSEEN's solution, makes practically the same criticism*: "I cannot see myself that OSEEN has proved his point. It is doubtless true that a great distance between the planes is unfavourable to stability, but to arrive at a sure conclusion there must be no limitation upon the character of the infinitesimal disturbance, whereas (as it appears to me) OSEEN assumes that the disturbance does not sensibly reach the walls. The simultaneous evanescence at the walls of both velocity-components of an otherwise sensible disturbance would seem to be of the essence of the question."

Method of OSBORNE REYNOLDS.

10. An entirely new line of attack was adopted by OSBORNE REYNOLDS in a paper published in 1895.† We shall illustrate his method in reference to disturbances of two-dimensional type, for which the governing equation is

$$\frac{\partial \zeta}{\partial t} - \nu \nabla^2 \zeta + (U + u') \frac{\partial \zeta}{\partial x} + v' \frac{\partial \zeta}{\partial y} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)\ bis$$

There is no necessity, in this method, to restrict the disturbance to be infinitesimal. In (12) we substitute $-\dfrac{\partial \psi}{\partial y}$ for u', $\dfrac{\partial \psi}{\partial x}$ for v', and $\nabla^2 \psi$ for ζ. If then we multiply the equation by ψ and integrate throughout the fluid field, terms which are of the second order make no resultant contribution, and we obtain

$$\tfrac{1}{2} \frac{\partial}{\partial t} \iint \left\{ \left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right\} dx\, dy = \beta \iint \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y}\, dx\, dy - \nu \iint (\nabla^2 \psi)^2\, dx\, dy. \quad . \quad . \quad . \qquad (23)$$

This equation enables us to calculate, when the distribution of ψ at any instant is known, the rate at which the integral

$$\tfrac{1}{2} \iint \left\{ \left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right\} dx\, dy$$

is increasing at that instant. This integral may also be written as $\frac{1}{2} \iint (u'^2 + v'^2)\, dx\, dy$,

* 'Collected Papers,' vol. 6, p. 272 (1914).

† 'Phil. Trans.,' A, vol. 186, p. 123 (1895).

where u' and v' define the disturbance according to (6); and for this reason it has been somewhat loosely termed the "kinetic energy integral of the disturbance." If it ultimately comes to zero, u' and v' must evidently come to zero at every point in the field, —that is to say, the steady motion is stable for the type of disturbance considered: if it can increase continually, we shall be justified in asserting that the steady motion is unstable.

Reynolds does not seem to have applied his method to the problem which we are discussing here; but for other types of steady motion,* by taking arbitrarily chosen types of disturbance which satisfy the boundary conditions, he was able to deduce a limiting velocity which must be exceeded if the kinetic energy integral is not to decrease. The criterion is expressed as a limiting value of a non-dimensional quantity akin to the quantity Vd/ν which we have taken as the "Reynolds' number" of our problem.† Numbers of this kind are known from experiment to be criteria of critical velocity in all cases where instability has been found to occur; and Reynolds seems to have concluded from this fact that his calculations had some bearing on the basic question of stability.

The application of Reynolds' method to our problem was made by H. A. Lorentz,‡ who obtained the limiting figure 288 for Vd/ν, taking as his assumed disturbance a species of "elliptic whirls."

11. But if the type is chosen arbitrarily (as in these investigations), the rate at which the integral is found to increase will as a matter of fact afford no indication of stability or instability. As remarked by G. I. Taylor§: "It does not determine an upper limit to the speed of flow which must be stable, because some other type of disturbance might exist which would increase initially at a lower speed of the fluid. Neither does it determine a lower limit to the speeds at which the flow must be unstable, because the assumed disturbance which initially increases might decrease indefinitely at some later stage of the motion."

Orr's *Extension of the Method.*

12. A considerable improvement in method has been effected by Orr, who employed the calculus of variations to find that type of disturbance in which, for the least value of β, the kinetic energy integral can be momentarily stationary. In this way he determined the highest value of Reynolds' number for which *all* disturbances must necessarily decrease. This value is given approximately by

$$R = 177. \qquad (24)$$

* See Introduction, § 3.

† See Introduction, § 8.

‡ 'Abhandlungen über theoretische Physik,' vol. 1, p. 43 (1907).

§ 'Phil. Trans.,' A, vol. 223, p. 290 (1923).

13. A paper published by one of us in 1924* extended ORR's results by finding a lower limit for the value of REYNOLDS' number which permits the kinetic energy integral to increase momentarily at any specified rate. That is to say, assuming the relation

$$\frac{d}{dt}\iint\left\{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right\}dx\,dy$$
$$= -2q\iint\left\{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right\}dx\,dy, \quad . \quad . \quad (25)$$

the paper investigated what is the smallest value of REYNOLDS' number which can be associated with any given value of q.

Fig. 3 shows the relation which was obtained. q has been rendered non-dimensional by associating it with other parameters of the system, so that abscissæ in fig. 3 represent not q but qd^2/ν. For $q = 0$ the curve gives ORR's result (24).

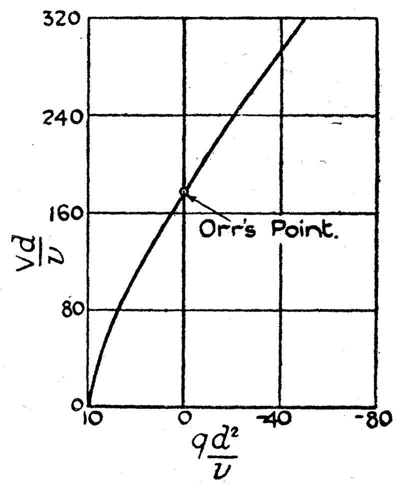

FIG. 3.

But results of this nature have little bearing on the question of stability, for reasons indicated by TAYLOR in the sentence quoted above. As a means of examining the question of stability it appears that the method of OSBORNE REYNOLDS must be discarded.

SECTION II.—*Review of Available Methods*: (1) *The Use of an Integral Equation.*

The "Vorticity Integral."

1. To obtain equation (23) of the previous section, which introduced the "kinetic energy integral," we multiplied the governing equation†

$$\frac{\partial\zeta}{\partial t} - \nu\nabla^2\zeta + (U + u')\frac{\partial\zeta}{\partial x} + v'\frac{\partial\zeta}{\partial y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

by ψ, and integrated throughout the fluid field, making use of the boundary conditions

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we multiply (1) by ζ instead of ψ, the term in U vanishes on integration (since U

* 'Phil. Mag.,' vol. 48, p. 540 (1924).

† Equation (12) of Section I.

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is independent of x), and the last two terms make no resultant contribution, in virtue of the equation of continuity. We are left with the equation

$$\frac{\partial}{\partial t}\iint \zeta^2\, dx\, dy = 2\nu \iint \zeta \nabla^2 \zeta\, dx\, dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

2. Since

$$\iint \zeta \nabla^2 \zeta\, dx\, dy = \int \zeta \frac{d\zeta}{\partial n} ds - \iint \left\{ \left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 \right\} dx\, dy, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

by GREEN'S theorem, equation (3) would yield an immediate proof of stability *if the boundary conditions had required either* ζ *or* $\partial\zeta/\partial n$ *to vanish, as well as* u', *at every point on the boundaries.* For under these conditions the line integral in (4) would vanish, and then (3) would show that $\iint \zeta^2\, dx\, dy$, which we shall term the "vorticity integral," decreases without limit. This means that ζ ($= \nabla^2\psi$) must ultimately come to zero at all points in the field, and hence (since $\partial\psi/\partial y$ vanishes at every point on the boundaries) ψ must ultimately have a constant or zero value throughout the field. That is to say, every disturbance will ultimately disappear.

Attempt to Deduce Stability from KORTEWEG'S *Theorem.*

3. Although this argument is not admissible (because the boundary conditions do not require either ζ or $\partial\zeta/\partial n$ to vanish), the fact that U does not appear in (3) seemed at one time to indicate that a proof of stability could be based on this equation. When the boundaries are fixed (so that $U = 0$), KORTEWEG'S theorem* asserts that the vorticity integral tends always to decrease, whatever the type of the disturbance, *provided that this is sufficiently small to justify neglect of the second-order terms in the equations of motion.* But equation (3), which *holds for disturbances of any magnitude,* indicates that the increase or decrease of the vorticity integral turns solely upon the type of the disturbance (which determines the sign of $\iint \zeta\nabla^2\zeta\, dx\, dy$), and does not depend either upon the scale of the disturbance or upon the magnitude of U. Hence it appeared that the integral $\iint \zeta\nabla^2\zeta\, dx\, dy$, since it is negative under conditions contemplated in KORTEWEG'S theorem, must be negative for every disturbance which satisfies the boundary conditions of our problem. If this conclusion were correct, equation (3) could be used to show that the vorticity integral tends steadily to decrease for any two-dimensional "free" disturbance, whether large or small, at a rate which is independent of U. It would follow that ζ must ultimately vanish at every point in the field, which means (as before) that the disturbance must ultimately disappear.

* 'Phil. Mag.,' vol. 16, p. 112 (1883). *Cf.* LAMB, "Hydrodynamics," §§ 329, 344 (2).

Failure of the Attempt.

4. Attention was accordingly directed to the sign of the integral $\iint \zeta\nabla^2\zeta\, dx\, dy$ for a disturbance governed by the boundary conditions (2). An alternative argument was developed, based on the assumption that any disturbance can be expanded in a series of normal functions of a certain class, which indicated that the integral is intrinsically negative for all disturbances satisfying the boundary conditions of our problem.

It was recognized that the necessity of assuming the validity of the expansion laid this demonstration open to some objection; but it was thought that, combined with the alternative argument from KORTEWEG'S theorem, it might be held to dispose fairly satisfactorily of the question of stability. Subsequently, however, in an attempt to develop a more convincing proof, it was found that certain types of disturbance, satisfying the boundary conditions (2), give a positive value to the integral $\iint \zeta\nabla^2\zeta\, dx\, dy$. If we consider a disturbance of the type

$$\psi = \mathrm{Y} \cos kx, \quad \ldots \ldots \ldots \ldots \quad (5)$$

where Y is a function of y only which satisfies the boundary conditions

$$\mathrm{Y} = d\mathrm{Y}/dy = 0, \quad \ldots \ldots \ldots \ldots \quad (6)$$

then the sign of $\iint \zeta\nabla^2\zeta\, dx\, dy$ is the sign of

$$\int (\mathrm{Y}'' - k^2\mathrm{Y})(\mathrm{Y}'''' - 2k^2\mathrm{Y}'' + k^4\mathrm{Y})\, dy,$$

$$= \int \mathrm{Y}''\mathrm{Y}''''\, dy - k^2 \int [\mathrm{Y}(\mathrm{Y}'''' - 3k^2\mathrm{Y}'') + 2\mathrm{Y}''^2 + k^4\mathrm{Y}^2]\, dy. \quad \ldots \quad (7)$$

(Dashes here denote differentiations with respect to y.) Now in virtue of the boundary conditions (6) we have

$$\int \mathrm{Y}\,\mathrm{Y}''''\, dy = \int \mathrm{Y}''^2\, dy,$$

and

$$\int \mathrm{Y}\,\mathrm{Y}''\, dy = -\int \mathrm{Y}'^2\, dy.$$

So the second integral in (7) is a necessarily positive quantity; and it follows that the sign of $\iint \zeta\nabla^2\zeta\, dx\, dy$, for a disturbance of the type (5), will be negative if the first integral $\int \mathrm{Y}''\,\mathrm{Y}''''\, dy$ can be shown to be negative. Since, on the other hand, the second integral in (7) vanishes with k, the first integral must predominate when k is sufficiently small, and hence $\iint \zeta\nabla^2\zeta\, dx\, dy$ may assume a positive value if $\int \mathrm{Y}''\,\mathrm{Y}''''\, dy$

can be positive. Thus to prove that $\iint \zeta \nabla^2 \zeta \, dx \, dy$ is negative, *it is both necessary and sufficient* to prove that $\int Y'' Y'''' \, dy$ is negative when Y is subject to the boundary conditions (6).

But if we define Y by the differential equation

$$Y'''' = y^{2n}, \qquad (8)$$

a solution even in y can be obtained in the form

$$Y = \frac{y^{2n+4}}{(2n+4)(2n+3)(2n+2)(2n+1)} + Ay^2 + B, \qquad (9)$$

where A and B are arbitrary constants. If the origin is taken at the centre of the field, so that the boundaries are defined by $y = \pm b$, the conditions (6) require that A shall have the value

$$\frac{-b^{2n+2}}{2(2n+3)(2n+2)(2n+1)},$$

and B a value which does not concern us here. Inserting this value for A in (9), we deduce that

$$Y'' = \frac{(2n+3)y^{2n+2} - b^{2n+2}}{(2n+3)(2n+2)(2n+1)}, \qquad (10)$$

and from (8) and (10) we have

$$\int Y'' Y'''' \, dy = \int_{-b}^{b} \frac{(2n+3)y^{4n+2} - b^{2n+2}y^{2n}}{(2n+3)(2n+2)(2n+1)} \, dy,$$

$$= \frac{8n(n+1)b^{4n+3}}{(2n+3)(2n+2)(2n+1)^2(4n+3)}.$$

Hence $\int Y'' Y'''' \, dy$ is positive when $n > 0$.

Reason for the Failure. Alternative Proof of KORTEWEG'S *Theorem.*

5. It thus became clear that $\int \zeta \nabla^2 \zeta \, dx \, dy$ is not intrinsically negative for every arbitrarily chosen distribution of ψ which satisfies the boundary conditions (2); and hence, that the reason why it is always negative according to KORTEWEG'S theorem is to be sought in some limitation, other than these boundary conditions, which is there imposed on ψ. It must be recognised that a *temporary* increase in the vorticity integral is not *a priori* incompatible with *ultimate* evanescence of all disturbances (*i.e.*, with stability). An alternative proof of KORTEWEG'S theorem will now be given, which indicates that we may expect such temporary increase to occur even when $U = 0$ (so that the boundaries are fixed), unless the disturbance is of infinitesimal magnitude.

In the circumstances contemplated by Korteweg, $U = 0$ and (since the disturbance is assumed to be infinitesimal) the second order terms in (1) may be neglected. So the governing equation is

$$\frac{\partial \zeta}{\partial t} = \nu \nabla^2 \zeta, \quad \ldots \ldots \quad (11)$$

where $\zeta = \nabla^2 \psi$. The boundary conditions (2) may be replaced by

$$\psi = \frac{\partial \psi}{\partial y} = 0,$$

for reasons which are given, in support of the same conclusion, in Section III, § 1.

Now by Green's theorem, used in conjunction with these boundary conditions which must be satisfied by ψ *at all times*, we have the equation

$$\iint \phi \frac{\partial \zeta}{\partial t} dx\, dy = \iint \phi \,.\, \nabla^2 \frac{\partial \psi}{\partial t} dx\, dy = \iint \frac{\partial \psi}{\partial t} \,.\, \nabla^2 \phi \, dx\, dy,$$

when ϕ is any continuous function. From this equation, and from (11), it follows that

$$\nu \iint \theta \nabla^2 \zeta \, dx\, dy = 0, \quad \ldots \ldots \quad (12)$$

when θ *is any plane harmonic function* (so that $\nabla^2 \theta = 0$), provided that ζ relates to an infinitesimal "free" disturbance and that the boundaries are at rest.

On this understanding, (12) permits us to write

$$\nu \iint \zeta \nabla^2 \zeta \, dx\, dy = \nu \iint (\zeta - \theta) \nabla^2 (\zeta - \theta) \, dx\, dy, \quad \ldots \ldots \quad (13)$$

and by Green's theorem, if θ be now made equal to ζ at every point on the boundaries, we have

$$\iint (\zeta - \theta) \nabla^2 (\zeta - \theta) \, dx\, dy = -\iint \left\{ \left(\frac{\partial}{\partial x}(\zeta - \theta)\right)^2 + \left(\frac{\partial}{\partial y}(\zeta - \theta)\right)^2 \right\} dx\, dy.$$

The right-hand side of this equation will clearly be negative unless $\zeta = \theta$ everywhere, so that $\nabla^4 \psi = 0$. But this result, in virtue of the boundary conditions, would mean that the disturbance vanished everywhere: so we may conclude from (13) that $\iint \zeta \nabla^2 \zeta \, dx\, dy$ is negative for all disturbances, under the conditions assumed in deriving (11) from (1). Thus Korteweg's theorem is established.

If on the other hand we dispensed with Korteweg's assumptions, and worked from the exact equation (1) in place of the simplified equation (11), we should obtain, in place of (12), the equation

$$\iint \theta \left[\nu \nabla^2 \zeta - (U + u') \frac{\partial \zeta}{\partial x} - v' \frac{\partial \zeta}{\partial y} \right] dx\, dy = 0, \quad \ldots \ldots \quad (14)$$

when θ is any plane harmonic function,—provided that the boundary conditions (2) can be modified in the same manner as before.

In virtue of the boundary conditions, and of the equation of continuity

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0,$$

we may throw this equation into the equivalent form*

$$\iint\left[\nu\theta\nabla^2\zeta+\zeta\left\{(\mathrm{U}+u')\frac{\partial\theta}{\partial x}+v'\frac{\partial\theta}{\partial y}\right\}\right]dx\,dy=0, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and we have further

$$\iint\zeta\mathrm{U}\frac{\partial\theta}{\partial x}\,dx\,dy=\iint\psi\nabla^2\left(\mathrm{U}\frac{\partial\theta}{\partial x}\right)dx\,dy,$$

by GREEN'S theorem, combined with (2),

$$=2\iint\psi\frac{d\mathrm{U}}{dy}\frac{\partial^2\theta}{\partial x\,\partial y}\,dx\,dy$$

(since U and θ are both plane harmonic),

$$=2\beta\iint\psi\frac{\partial^2\theta}{\partial x\,\partial y}\,dx\,dy.$$

Substituting this expression in (15), we should have, in place of (13), the equation

$$\nu\iint\zeta\nabla^2\zeta\,dx\,dy=\nu\iint(\zeta-\theta)\,\nabla^2\,(\zeta-\theta)\,dx\,dy$$
$$-\iint\left[2\beta\psi\frac{\partial^2\theta}{\partial x\,\partial y}+\zeta\left(u'\frac{\partial\theta}{\partial x}+v'\frac{\partial\theta}{\partial y}\right)\right]dx\,dy. \quad . \quad . \quad . \quad (16)$$

It is not apparently possible to deduce from this relation that the integral on the left is in all cases negative, unless we make β zero and assume that the disturbance is of infinitesimal magnitude. Moreover, it is only on this last assumption that it seems possible to justify the modification of the boundary conditions which has led to (14).

Physical Explanation of the Failure. "Boundary Conduction."

6. The foregoing paragraph shows that there is operative, in addition to (2), a further boundary condition which assumes a different form according as KORTEWEG'S restrictions are imposed or not. This additional condition is expressed by (14), which under KORTEWEG'S assumptions reduces to (12). In either form it has been based on the consideration that not only ψ and $\partial\psi/\partial y$, but also $\dot{\psi}$ and $\partial\dot{\psi}/\partial y$, must vanish at the

* U is, of course, independent of x.

boundaries throughout all time; for the latter requirement demands that $\iint \theta\dot{\zeta}\, dx\, dy$ shall vanish when θ is *any* plane harmonic function, and for "free" disturbances, by substituting for $\dot{\zeta}$ from (1), we arrive at (14). So long as the disturbance is "*forced*" (so that body forces appear in the governing equation), the restriction is not operative; but it is imposed the moment the disturbance becomes "*free*," and in general it demands a conduction of vorticity (ζ) inward from the boundary. Therefore at the boundary $\dot{\zeta}$, and hence* $\nabla^2\zeta$, considered as functions of time, may exhibit discontinuities.

7. A similar conclusion, in explanation of our paradox, is reached by LAMB in a paper† which deals with the simpler case of a laminar disturbance ($v = 0$) occurring in a channel with fixed walls. LAMB assumes an initial distribution of velocity (u) to be produced by body forces of simple type; and he represents this by a Fourier series, each term of which satisfies the boundary condition ($u = 0$) and contains a time factor which makes it a solution of the governing equation for "free" disturbances. Finally, using this series to examine the history of the disturbance after the body forces have ceased to act, he finds that $\partial^2 u/\partial y^2$ and higher derivatives of u are discontinuous in respect of time.

According to the argument now given, we must look to conduction of vorticity from the boundaries into the fluid field as the only means whereby the disturbance can be made to satisfy a third boundary condition, implicit in the governing equation for *free* disturbances. This, we suggest, is the physical explanation of both LAMB's result and ours.

Further Examination of the Integral Equation. (1) RAYLEIGH'S *Analogies.*

8. The foregoing discussion, in its insistence on the importance of boundary conduction, supports the objections advanced by RAYLEIGH and ORR against investigations (by KELVIN and OSEEN) which in effect make the question of stability turn on the history of a disturbance in fluid of infinite extent.‡ But it also calls for a critical examination of equation (3), which appears to have been accepted both by ORR§ and by RAYLEIGH.|| RAYLEIGH remarked that the governing equation for ζ will apply equally to the temperature, or salinity, of a fluid moving with velocity U; and he added: "Any conclusions that we may draw have thus a widened interest." But, in our hydrodynamic problem, the subsequent distribution of ζ is controlled not only by the governing equation but also by the condition $\left(\iint \theta\dot{\zeta}\, dx\, dy = 0\right)$. This is a highly artificial condition when ζ is

* In virtue of the governing equation.

† 'Aeronautical Research Committee' R. & M., No. 1084 (1926).

‡ See Section I, §§ 8 and 9.

§ *Loc. cit. supra,* end of § 23.

|| 'Collected Papers,' vol. 6, p. 270 (1914).

interpreted as temperature or salinity: the natural condition in such cases would be either $\zeta = 0$ or $\partial\zeta/\partial y = 0$ at the boundaries throughout all time, and neither of these conditions would introduce the difficulties which confront us here, as a consequence of boundary conduction.

(2) *The Case of Vanishing Viscosity* ($\nu = 0$).

9. Before concluding our discussion of equation (3), we have to make one last remark. If the fluid were completely inviscid ($\nu = 0$), the second term in (1) would disappear, and on multiplying (1) by ζ and integrating throughout the fluid field we should obtain

$$\frac{\partial}{\partial t}\iint \zeta^2\, dx\, dy = 0, \qquad (17)$$

—an equation which could also have been obtained by putting ν zero in (3).

The interpretation of (17) is that, when the fluid is inviscid, our steady motion possesses neutral stability, in the sense that a disturbance can neither disappear nor indefinitely increase. This conclusion was to be expected on physical grounds, since in the absence of viscosity there is no mechanism whereby the rotation of any fluid element can be changed*; no complication arises from boundary conduction, since this again can operate only in virtue of viscosity. To describe the stability as neutral seems appropriate, although the consideration that all the vorticity might (so far as equation (17) is concerned) subsequently be concentrated in a finite portion of the field, and so attain great local intensity, illustrates the difficulty (remarked by RAYLEIGH†) of framing an accurate definition of what we mean by stability.

RAYLEIGH'S *Researches on the Stability of Inviscid Fluids.*

10. Equation (17), which we have just shown to be valid for inviscid fluids, has an important bearing on numerous researches in which RAYLEIGH sought to avoid the difficulties associated with the viscosity term in (1) by assuming viscosity to be operative in determining the type of the steady motion, but without effect on the subsequent history of an arbitrary disturbance. In other words, RAYLEIGH simplified (1) by omitting the term $\nu\nabla^2\zeta$, and then sought a solution on the assumption that U is a specified function of y. A number of functions were tried,—among them the linear function which is the actual form of U in the problem of this paper. RAYLEIGH'S conclusion was that the steady motion is stable or unstable (for a fluid of vanishing viscosity) according as U, regarded as a function of y, has a curvature which is or is not of one sign throughout the breadth of the field. In the nature of the case, no question of a critical speed (or value of REYNOLDS' number) arises.

* Equation (1) with the term in ν suppressed may in fact be regarded as expressing the constancy of ζ for a particular element of the fluid.

† 'Collected Papers,' vol. 6, p. 202 (1913).

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11. A linear distribution of U may be regarded (since the "curvature" of U is everywhere zero) as a limiting case in which RAYLEIGH's criterion leads to the conclusion of neutral stability; and in this sense his result supports the conclusion which we have drawn from (17). But it may be questioned whether investigations on the lines proposed by RAYLEIGH can be expected to lead to results of any significance. Whatever be the form of U (as a function of y), equation (17) can be obtained; and if this equation is admitted, *neutral stability may be inferred for any laminar steady motion,* since equation (17) shows that the average value of ζ^2 is invariant in respect of time.

It thus becomes necessary to consider the question why, since (17) indicates neutral stability for all types of steady motion, RAYLEIGH should have obtained any criterion of stability from his investigations. The explanation of this paradox would appear to be as follows:—Equation (17) has been deduced from (1) after suppression of the viscosity terms; and equation (1) has been derived by assuming two solutions of the fundamenta lhydrodynamic equations to exist,—namely, a steady motion ($u = \mathrm{U} = f(y)$; $v = 0$) and a disturbed motion ($u = \mathrm{U} + u'$; $v = v'$). Subtracting the two solutions, we have obtained equations which govern the disturbance u', v'; and on account of the form of the original equations these derived equations do not involve the body forces. But this means, physically, *not that the body forces are assumed to become inoperative when the disturbance becomes "free," but that they are assumed to remain unchanged.* Therefore, when we take a definite function of y to represent the steady motion U and insert this in the equation which governs u', v', we are investigating the subsequent history of a disturbance which develops, *not under no body forces, but under that system of body forces which would be required to bring* U *into existence.*

12. Suppose that the body forces do not form a conservative system. Then a fluid element, by following some suitable path, can take up energy from them; and whilst energy will not be absorbed in this way so long as the motion is laminar, it may be so absorbed in a disturbance of suitable type. The condition that body forces confined to the direction of x (such as are required to establish laminar motion) shall be "conservative" is $\partial \mathrm{X}/\partial y = 0$, or (in the circumstances which are here contemplated) $\mathrm{X} = \text{const.}$ Under this condition the equation for the steady motion is

$$\nu \frac{d^2\mathrm{U}}{dy^2} = \text{const.},$$

which shows (since ν cannot change sign) that the curvature represented by $d^2\mathrm{U}/dy^2$ must be of one sign throughout the breadth of the field, if the velocity U has been built up by a conservative system of body forces.

It thus appears that in those types of steady motion considered by RAYLEIGH in which $d^2\mathrm{U}/dy^2$ changed sign in the field (and for which he deduced instability), he was tacitly assuming the operation of a non-conservative system of body forces, and investigating the subsequent history of a disturbance developing under the action of that system. It is therefore not surprising that he should have deduced a possibility of hydrodynamic instability.

SECTION III.—*Review of Available Methods:* (2) *The Method of Expansion in Normal Co-ordinates.*

1. The method of normal co-ordinates is applicable to a large number of physical problems, but it is restricted to cases in which the governing equation is linear. This restriction, in our problem, confines attention to infinitesimal disturbances: the governing equation then becomes

$$\frac{\partial \zeta}{\partial t} - \nu \nabla^2 \zeta + U \frac{\partial \zeta}{\partial x} = 0, \quad \ldots \quad (1)$$

where

$$\zeta = \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} = \nabla^2 \psi, \quad \ldots \quad (2)$$

and the conditions at the boundaries are

$$v' = \frac{\partial \psi}{\partial x} = 0, \qquad -u' = \frac{\partial \psi}{dy} = 0. \quad \ldots \quad (3)^*$$

To explain the method, we may take as before the special case in which $U = 0$, so that equation (1) becomes

$$\frac{\partial \zeta}{\partial t} - \nu \nabla^2 \zeta = 0. \quad \ldots \quad (4)$$

It will be observed that equation (1) reduces to this form, whatever the value of U, provided that the disturbance is laminar (so that ψ is independent of x).

At any particular instant in its history, the disturbance may be analysed into a series of component disturbances, each of which is harmonic in x with some particular wave-length; and since the governing equation is linear these component disturbances will behave independently of one another throughout all time. Stability for laminar disturbances has been proved.† Therefore in discussing either (1) or (4) we may assume that the disturbance is harmonic in x, and then the first of (3) may be replaced by the condition that ψ must vanish at all points on the boundary, so that in place of (3) we may write

$$\psi = \frac{\partial \psi}{\partial y} = 0, \text{ at the boundaries.} \quad \ldots \quad (5)$$

* Equations (1), (2) and (3) are equations (14), (13) and (11), respectively, of Section I.

† See § 5 of Section I.

(A) *Application to the special case* (U = 0). "*Normal*" *Solutions.*

2. The basic assumption of the method is that any solution of the governing equation (compatible with the boundary conditions) may be analysed into a series of component solutions, each of which is a simple exponential function (with real or imaginary index) in respect of time. These component solutions are termed "normal" solutions, or co-ordinates; if we can determine the nature of their variation with time (*i.e.*, whether any tend to increase), then, according to the assumption stated, we can decide the question of stability for the most general type of disturbance that can occur.

The justification of the assumption (which is the basis of LAGRANGE's general treatment of dynamical problems) would appear to come from an argument by induction. If we were dealing with a system characterised by a finite number n of degrees of freedom, the equations of motion would also number n; they would correspond with the equation (4) which governs our continuous system at every point in the field. Examining the conditions for a solution of "normal" type, we should find that n (in general) distinct solutions of this type exist,—that is, exactly the number required to enable us to represent any possible configuration of the system as a series of normal solutions. The assumption is therefore justified when n has any finite value however great; and it is inferred that a corresponding property may be postulated for systems in which n is infinite,—that is, for continuous systems such as we are considering here. This argument, of course, has no pretensions to mathematical rigour. But at least we may say that a series of n normal solutions can be found which will represent any specified disturbance at n points, and that no finite limit is imposed on the magnitude of n. If then all of these normal disturbances have negative time factors, the question of instability is relegated to a type of disturbance (namely, the difference between the specified disturbance and the disturbance actually represented by our series) which on physical grounds may be regarded as most unlikely to persist.

3. Making the assumption, we assert that any solution of (4) may be represented by a series of the form

$$\psi = A_1\psi_1 + A_2\psi_2 + \ldots + A_n\psi_n + \ldots, \qquad (6)$$

where, for example, ψ_n has the form

$$\psi_n = e^{\lambda_n t}\Psi_n, \qquad (7)$$

so that Ψ_n is a solution of the "normal equation"

$$[\lambda_n\nabla^2 - \nu\nabla^4]\Psi_n = 0 \qquad (8)$$

which satisfies the boundary conditions

$$\Psi_n = \frac{\partial\Psi_n}{\partial y} = 0. \qquad (9)$$

The values of $\lambda_1 \ldots \lambda_n \ldots$ are at present unknown, and are not, *a priori*, necessarily real. If we can show that they are all negative or have their real parts negative, then, according to (6), any possible solution of (4) must ultimately come to zero. Hence, to decide the question of stability when $U = 0$ it will be sufficient to confine attention to equations (8) and (9). The question, how to determine the coefficients $A_1 \ldots$, etc. in (6), will still be open; but this will not matter if we are prepared to assume that the series exists.

The Conjugate Property.

4. To investigate the nature of $\lambda_1 \ldots$, etc., we make use of "conjugate relations" which can be shown to hold between any two normal solutions of equation (8).

Let Ψ_m be a second normal solution, satisfying the equations which are obtained when m is written for n in (8) and (9). Multiply (8) by Ψ_m, and integrate throughout the fluid field: then we have

$$\lambda_n \iint \Psi_m \nabla^2 \Psi_n \, dx \, dy = \nu \iint \Psi_m \nabla^4 \Psi_n \, dx \, dy, \quad \ldots \ldots \quad (10)$$

and in exactly the same way we may obtain the equation

$$\lambda_m \iint \Psi_n \nabla^2 \Psi_m \, dx \, dy = \nu \iint \Psi_n \nabla^4 \Psi_m \, dx \, dy. \quad \ldots \ldots \quad (11)$$

But, in virtue of the boundary conditions which are satisfied both by Ψ_m and Ψ_n, we have

$$\left.\begin{aligned} &\iint \Psi_m \nabla^2 \Psi_n \, dx \, dy = \iint \Psi_n \nabla^2 \Psi_m \, dx \, dy \\ \text{and} \quad &\iint \Psi_m \nabla^4 \Psi_n \, dx \, dy = \iint \Psi_n \nabla^4 \Psi_m \, dx \, dy. \end{aligned}\right\} \quad \ldots \ldots \quad (12)$$

Hence we may deduce from (10) and (11) that

$$(\lambda_m - \lambda_n) \iint \Psi_m \nabla^2 \Psi_n \, dx \, dy = 0,$$

and it follows from (10) that

$$\iint \Psi_m \nabla^2 \Psi_n \, dx \, dy = \iint \Psi_m \nabla^4 \Psi_n \, dx \, dy = 0, \text{ if } \lambda_m \neq \lambda_n. \quad \ldots \quad (13)$$

These are the "conjugate relations" which hold between "normal" solutions.

Reality of the Time Factor in a "Normal" Solution.

5. The relations (13) may be used to show that (in this special case of our problem) all the λ's are real. For if in (8) λ_n were complex or imaginary, so that we could write

$$\lambda_n = \alpha + i\beta, \quad \text{where } \alpha \text{ and } \beta \text{ are real,}$$

then Ψ_n would also be complex or imaginary, so that we should have

$$\Psi_n = R + iI,$$

where R and I are real functions of x and y.

It is easy to verify that a second solution $\overline{\Psi}_n$ would exist, associated with a constant $\overline{\lambda}_n$, where

$$\overline{\Psi}_n = R - iI$$

and

$$\overline{\lambda}_n = \alpha - i\beta.$$

Hence, writing $\overline{\Psi}_n$ for Ψ_m in (13), we see that, *unless* $\overline{\lambda}_n = \lambda_n$,

$$\iint \overline{\Psi}_n \nabla^4 \Psi_n \, dx\, dy = 0.$$

But

$$\iint \overline{\Psi}_n \nabla^4 \Psi_n \, dx\, dy = \iint \nabla^2 \overline{\Psi}_n \, . \, \nabla^2 \Psi_n \, dx\, dy,$$

$$= \iint \{(\nabla^2 R)^2 + (\nabla^2 I)^2\} \, dx\, dy,$$

and the last integral cannot vanish unless $\nabla^2 R$, $\nabla^2 I$, and therefore (by the boundary conditions) R and I, vanish everywhere. We deduce that $2i\beta$ $(= \lambda_n - \overline{\lambda}_n)$ must be zero, so that λ_n and $\overline{\lambda}_n$ are purely real.

All "Normal" Disturbances come ultimately to zero.

6. Given this result, we can show at once that all the λ's are in fact negative in the special case now considered. Multiplying (8) by Ψ_n and integrating, we have

$$\lambda_n \iint \Psi_n \nabla^2 \Psi_n \, dx\, dy = \nu \iint \Psi_n \nabla^4 \Psi_n \, dx\, dy, \quad \ldots \ldots \quad (14)$$

or, by GREEN's theorem, in virtue of the boundary conditions (9),

$$-\lambda_n \iint \left\{ \left(\frac{\partial \Psi_n}{\partial x}\right)^2 + \left(\frac{\partial \Psi_n}{\partial y}\right)^2 \right\} dx\, dy = \nu \iint (\nabla^2 \Psi_n)^2 \, dx\, dy. \quad \ldots \ldots \quad (15)$$

Since both integrals are necessarily positive, this proves the required result.

The Method of Expansion.

7. Thus, on our starting assumption, we have demonstrated stability (in the case where $U = 0$, and the disturbance is of infinitesimal magnitude) without following out the detailed history of any disturbances other than those of "normal" type. To determine completely the history of a given disturbance, we should in general have

required to know the value of $A_1 \ldots$, etc. in the expansion (6),—that is, actually to perform the expansion. It is a great advantage of the method that this procedure is not necessary; but on the assumption that the expansion exists, the conjugate relation does in fact provide us with a method for determining the coefficients.

Initially (when $t = 0$), we have from (6) and (7)

$$\psi = \psi_0 = A_1\Psi_1 + \ldots + A_n\Psi_n + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Multiplying both sides of this equation by $\nabla^2\Psi_n$, and making use of the conjugate relations (13), we obtain on integration

$$\iint \psi_0 \nabla^2 \Psi_n \, dx \, dy = A_n \iint \Psi_n \nabla^2 \Psi_n \, dx \, dy, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

—a relation which serves to determine A_n when ψ_0 is given and the form of Ψ_n is known.

Approximate Determination of the Time Factor.

8. For some purposes it might have been desirable, not only to prove stability by showing that all the λ's are negative, but also to determine the numerically smallest value of λ,—that is, the rate of decay of the most persistent normal disturbance. (Evidently this component will predominate in any disturbance which has been "free" for a considerable time.) In other problems stability is replaced by instability, so that some of the λ's are positive; it then becomes important to determine the largest of these positive values, since the corresponding normal disturbance must ultimately predominate.

In the example which we are now considering, the λ's can be found without difficulty by exact methods. But in other systems an exact determination of λ, even for the most persistent disturbance, presents difficulties which are insuperable; and it is another important advantage of the "method of normal co-ordinates" that it provides a means of approximating to the λ's which does not involve an exact functional solution of the governing equation.

9. The procedure, which is due to RAYLEIGH, may be explained as follows:—Suppose that we calculate λ from the equation

$$-\lambda = \frac{\nu \iint (\nabla^2 \Psi)^2 \, dx \, dy}{\iint \left\{ \left(\frac{\partial \Psi}{\partial x}\right)^2 + \left(\frac{\partial \Psi}{\partial y}\right)^2 \right\} dx \, dy}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

by giving to Ψ any arbitrarily chosen distribution which does not violate the boundary conditions. Comparing (18) with (15), we see that (18) would be an exact value for λ_n

if our chosen distribution had agreed accurately with Ψ_n. We also see that (18) is equivalent to

$$\lambda = \frac{\nu \iint \Psi \nabla^4 \Psi \, dx \, dy}{\iint \Psi \nabla^2 \Psi \, dx \, dy}, \qquad (19)$$

and if in this equation we substitute for Ψ an expansion of the form (16), we have the expression

$$\lambda = \frac{\nu \iint \Sigma (A_n \Psi_n) \, . \, \Sigma (A_n \nabla^4 \Psi_n) \, dx \, dy}{\iint \Sigma (A_n \Psi_n) \, . \, \Sigma (A_n \nabla^2 \Psi_n) \, dx \, dy}.$$

Now this expression, in virtue of the conjugate relations (13), reduces to the form

$$\left.\begin{aligned} \lambda &= \frac{\nu \Sigma \left[A_n^2 \iint \Psi_n \nabla^4 \Psi_n \, dx \, dy \right]}{\Sigma \left[A_n^2 \iint \Psi_n \nabla^2 \Psi_n \, dx \, dy \right]}, \\ \text{or to} \qquad - \lambda &= \frac{\nu \Sigma \left[A_n^2 \iint (\nabla^2 \Psi_n)^2 \, dx \, dy \right]}{\Sigma \left[A_n^2 \iint \left\{ \left(\frac{\partial \Psi_n}{\partial x} \right)^2 + \left(\frac{\partial \Psi_n}{\partial y} \right)^2 \right\} dx \, dy \right]}. \end{aligned}\right\} \qquad (20)$$

So (20) gives the value which we shall obtain for λ when an arbitrarily chosen Ψ is inserted in (18),—A_1 ... , etc. being the constants in the series of type (16) by which this arbitrarily chosen Ψ is represented. Without seeking to determine A_1 ..., etc., we may deduce from (20) that *if* our assumed Ψ is in fact closely identical with a normal solution Ψ_1 (so that A_1 is large in comparison with all the other A's), then λ will be even more closely identical with λ_1. For $\lambda = \lambda_1$ when all the A's except A_1 are zero; and accordingly λ/λ_1 differs from unity by small quantities of the type $(A_n/A_1)^2$, —that is, by a small quantity of the second order.

Physical Analogues of "Normal" Solutions of the Simplified Equations.

10. Equations of type (8) and (9) are governing equations in other physical problems having no connection with Hydrodynamics. We may mention here the problem of elastic stability in a flat plate of uniform thickness subjected to uniform thrust in its plane, and the rather more artificial problem of flexural vibrations in a flat plate which has uniform "rotatory inertia," compared with which its ordinary ("transverse") inertia is negligible.

(B) *Application to the General Case.*

11. Turning now to the general case of our problem (where $U \neq 0$), we have to consider "normal" solutions of (1) and (5),—that is, solutions of the form (7). The normal equation (8) is now replaced by

$$\left[\lambda_n + U\frac{\partial}{\partial x} - \nu\nabla^2\right]\nabla^2\Psi_n = 0, \quad \ldots \ldots \ldots \ldots \quad (21)$$

which for some value of λ_n must be satisfied, together with the boundary conditions

$$\Psi_n = \frac{\partial\Psi_n}{\partial y} = 0, \quad \ldots \ldots \ldots \ldots \quad (9)\ bis$$

by a normal solution of our problem.

Failure of the Conjugate Relations.

12. In seeking to apply to the general case the methods which have been described above, we are at once confronted with the difficulty that solutions of (21) do not obey conjugate relations analogous to (13). Proceeding on the lines of § 4, we should obtain, in place of (10) and (11), two integral equations as follows :—

$$\left.\begin{aligned}\lambda_n\iint\Psi_m\nabla^2\Psi_n\,dx\,dy &= \nu\iint\Psi_m\nabla^4\Psi_n\,dx\,dy - \iint U\Psi_m\frac{\partial}{\partial x}\nabla^2\Psi_n\,dx\,dy,\\ \lambda_m\iint\Psi_n\nabla^2\Psi_m\,dx\,dy &= \nu\iint\Psi_n\nabla^4\Psi_m\,dx\,dy - \iint U\Psi_n\frac{\partial}{\partial x}\nabla^2\Psi_m\,dx\,dy.\end{aligned}\right\} \quad . \quad . \quad (22)$$

The relations (12) still hold ; but we have

$$\begin{aligned}\iint U\Psi_n\frac{\partial}{\partial x}\nabla^2\Psi_m\,dx\,dy &= \iint\frac{\partial\Psi_m}{\partial x}\nabla^2(U\Psi_n)\,dx\,dy,\\ &= \iint\frac{\partial\Psi_m}{\partial x}\left\{U\nabla^2\Psi_n + 2\frac{dU}{dy}\frac{\partial\Psi_n}{\partial y}\right\}dx\,dy\end{aligned}$$

(since U is linear in y and independent of x),

$$= -\iint\Psi_m\left\{U\frac{\partial}{\partial x}\nabla^2\Psi_n + 2\frac{dU}{dy}\frac{\partial^2\Psi_n}{\partial x\,\partial y}\right\}dx\,dy. \quad \ldots \ldots \ldots \ldots \quad (23)$$

Hence, on subtraction of the two equations (22) we obtain, using (12),

$$\begin{aligned}(\lambda_m - \lambda_n)\iint\Psi_m\nabla^2\Psi_n\,dx\,dy &= 2\iint\Psi_m\left\{U\frac{\partial}{\partial x}\nabla^2\Psi_n + \frac{dU}{dy}\frac{\partial^2\Psi_n}{\partial x\,\partial y}\right\}dx\,dy,\\ &= 2\iint\Psi_m\left\{\frac{\partial}{\partial x}\left(U\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(U\frac{\partial}{\partial y}\right)\right\}\frac{\partial\Psi_n}{\partial x}\,dx\,dy\,; \quad . \quad (24)\end{aligned}$$

and since the right-hand side of (24) does not of necessity vanish, we are unable to deduce relations of the type (13).

"Energetic" and "Non-Energetic" Terms.

13. We shall see the reason for this result if we throw (21) into the equivalent form

$$\lambda_n \nabla^2 \Psi_n = \nu \nabla^4 \Psi_n + \frac{dU}{dy}\frac{\partial^2 \Psi_n}{\partial x \partial y} - \left\{\left[\frac{\partial}{\partial x}\left(U\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(U\frac{\partial}{\partial y}\right)\right]\frac{\partial \Psi_n}{\partial x}\right\}, \quad . \quad . \quad (25)$$

in which, as before, β can be written for dU/dy. It is on account of the terms in twisted brackets that we obtain the integral on the right of (24): had they been absent, so that we had as the "normal equation"

$$\left[\lambda_n \nabla^2 - \nu \nabla^4 - \beta\frac{\partial^2}{\partial x\, \partial y}\right]\Psi_n = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

we could have deduced conjugate relations corresponding with (13).

Multiplying (25) by Ψ_n, and integrating throughout the fluid field, we obtain the integral equation

$$\lambda_n \iint \Psi_n \nabla^2 \Psi_n \, dx\, dy = \nu \iint (\nabla^2 \Psi_n)^2 \, dx\, dy + \beta \iint \Psi_n \frac{\partial^2 \Psi_n}{\partial x\, \partial y}\, dx\, dy. \quad . \quad . \quad (27)$$

But the same equation could have been derived, and by the same process, from (26). Hence we see that the terms in twisted brackets on the right of (25) make no contribution to the integral equation, when Ψ_n is taken as the multiplier.

Equation (27) may be termed the "energy integral equation," since the integral on the left is a measure of $\frac{1}{2}\iint (u'^2 + v'^2)\, dx\, dy$,—the "kinetic energy of the disturbance." It is in effect the equation discussed by OSBORNE REYNOLDS.* Accordingly we shall speak of the terms in twisted brackets as "convectional" or "non-energetic" (they do not affect the momentary rate of increase of the "kinetic energy"); the remaining terms in (25),—viz., the terms which appear in (26),—we shall describe as "energetic."

Non-energetic terms do not appear in the governing equations of ordinary problems in vibration theory. We shall show that their occurrence in our problem is the reason why the method of normal co-ordinates cannot be applied, at all events in its usual form.

Consequences of the Non-Energetic Terms.

14. The "non-energetic" terms, as we have seen, result in failure of the conjugate relations between normal solutions. If we turn back to §§ 5–9, we see that without

* Equation (23) of Section I.

the aid of these relations we could not have established that the λ's were necessarily real, or that their real parts were negative ; neither could we have justified RAYLEIGH'S approximate method for their calculation. These questions are accordingly still open in relation to the general case of our problem.

Had the non-energetic terms not been present, OSBORNE REYNOLDS' investigation would have had a real bearing on the question of stability ; in fact, it would have constituted an ordinary application of the method of normal co-ordinates (*cf.* § 6). Actually (since the Ψ which he inserts in (27) has an arbitrarily chosen distribution) it must be regarded as an application of RAYLEIGH'S method (§ 9) in circumstances where this method cannot be justified as a method of approximate calculation, because the conjugate relations do not hold.

Similar objections apply to any extension of REYNOLDS' method on the lines described in Section I, § 12, *if its results are regarded as an answer to the question of stability.* For the values of "λ" (denoted by $-q$ in Section I) are there deduced from distributions of Ψ which do not satisfy the correct normal equation (25), but the modified equation (26).

Résumé.

15. Summarising the conclusions of this section, we may say that the method of expansion in normal co-ordinates cannot be applied, at all events in its usual form, to the general case of our problem. The reason is that our normal equation, which may be written in the form (25), contains "non-energetic" terms which cause a breakdown of the conjugate relations. We defer for the present any consideration of the question whether the method can be modified to meet our requirements : what we have first to discuss is the question whether, because the conjugate relations fail, there is any necessity to discard the assumption that an arbitrary disturbance may be expanded in a series of normal solutions, satisfying equations of the type (25) and the boundary conditions (9).

Is the Expansion in Normal Co-ordinates Valid in the General Case?

16. A remark by RAYLEIGH* suggests that he had some doubt as to the validity of the expansion : "From his (KELVIN'S) results it appears that it is not possible to find a solution applicable to an unlimited fluid which shall be periodic with respect to x, and remain finite when $y = \pm \infty$, ... The cause of the failure would appear to lie in the fact, indicated by Lord KELVIN'S solution, that the stability is ultimately of a higher order than can be expressed by any simple exponential function of the time."

ORR† also considered this question in relation to our problem. He set himself to establish, (1) that the "frequency" equation (from which the λ's are to be found)

* 'Collected Papers,' vol. 4, p. 209 (1895).

† 'Proc. R. Irish Acad.,' vol. 27, pp. 9–138 (1907).

has an infinite number of roots, and (2) that the expansion, if valid, can be effected. It thus appears that he considered these to be necessary conditions for the validity of the expansion. He describes his investigation as "very incomplete and unsatisfactory,"* and found himself unable to perform the expansion.

In regard to RAYLEIGH's remark it may be contended that solutions applicable to an infinite field (*e.g.*, in the theory of elasticity or of the conduction of heat) have very little bearing on the question of expansion in normal co-ordinates when the field is limited and boundary conditions are imposed. In regard to the standpoint adopted by ORR it may be remarked that we have succeeded in finding a method of expansion,† but that this result does not seem to have much bearing on the question whether the expansion is in fact valid. In ordinary harmonic analysis, for example, there is no difficulty in determining values for the coefficients, but a relatively elaborate argument has to be employed to justify the expansion. In many physical problems for which the method of normal co-ordinates is employed, the validity of the expansion cannot be established rigorously, but is *assumed*,—presumably on the grounds of an argument by induction, of the kind which has been given in § 2. Now since this argument does not involve the conjugate relations, its conclusion is not invalidated merely because the conjugate relations do not hold. It seems reasonable to assume that the expansion is valid, at least to the extent that an arbitrary disturbance can be defined with sufficient accuracy by giving values to ψ at n points in the field, where n may be very large, but is finite.

17. It is almost certainly necessary to adopt this standpoint if we are to make any progress with our present problem, because we have seen (Sections I and II) that nothing definite seems to be obtainable by the use of integral equations. That is to say, we seem obliged to conclude that the steady motion is stable (in the sense that every infinitesimal disturbance comes ultimately to zero) if we can show that this equation has no solution for which the real part of λ is positive.

This is the standpoint adopted by most workers on our problem. RAYLEIGH‡ asserts that "a precise formulation of the problem for free infinitesimal disturbances was made by ORR (1907),"—and ORR's formulation was based on the "normal equation." His period-equation was given a little later (1908) independently by SOMMERFELD,§ whose paper stimulated the investigations of v. MISES|| and HOPF.¶ The latter do not appear to have been accepted by RAYLEIGH, although he agrees with their view that the motion is stable.**

* *Ibid.*, p. 74.

† This will be described in a subsequent paper.

‡ 'Collected Papers,' vol. 6, p. 274 (1914).

§ 'Atti Cong. Intern. Math.' (Roma, 1909), vol. 3, p. 116.

|| 'Jber. deuts. Math. Ver.,' vol. 21, p. 241 (1912).

¶ 'Ann. Physik,' vol. 44, p. 1 (1914).

** See Introduction, § 7.

18. In our own preliminary attack on the problem we decided that it would be preferable, instead of proceeding from the formal solution in terms of Bessel functions and substituting for these their semi-convergent approximations, to solve equation (21) by the laborious method of calculating series and combining them to satisfy the boundary conditions. Although this procedure was only expected to throw light on a limited range of REYNOLDS' number, it offered the advantage that the limits of error would be definitely known. Further, the same method would be applicable to other problems of hydrodynamic stability, in which the normal solutions cannot be expressed in terms of known functions.

This investigation is described in Section IV.

SECTION IV.—*Application of the Method of Normal Co-ordinates (Numerical Calculations).*

Non-Dimensional Form of the Governing Equation for "Normal" Disturbances.

1. For purposes of numerical calculation, the equation which governs a "normal" disturbance must be transformed so that the parameters of the problem, and the independent variable, appear in non-dimensional form.

This equation is obtained from (1) of Section III by assuming that ψ is harmonic in x and a simple exponential function of the time. Separating real and imaginary parts of the time factor, we may write

$$\psi = \Psi e^{-qt + i(pt + kx)} \qquad \ldots \ldots \ldots \ldots \quad (1)$$

where k, p and q are constants, and Ψ is a function of y only.

Now let $2b$ be the breadth (d) of the fluid field, and write

$$\left.\begin{aligned} &z \text{ for } ky, \\ &\gamma \text{ for } kb, \\ &\mathrm{P} \text{ for } p/\nu k^2 = pb^2/\nu\gamma^2, \\ &\mathrm{Q} \text{ for } q/\nu k^2 = qb^2/\nu\gamma^2, \\ &\mathrm{B} \text{ for } \beta/\nu k^2 = \beta b^2/\nu\gamma^2. \end{aligned}\right\} \qquad \ldots \ldots \ldots \ldots \quad (2)$$

Then the wave-length l of the disturbance, in the x-direction, is given by

$$l = 2\pi b/\gamma,$$

and REYNOLDS' number for our problem is defined by

$$\mathrm{R} = \mathrm{V}d/\nu = (2\beta b) \times (2b)/\nu = 4\gamma^2\mathrm{B}.$$

When modified by these substitutions, the governing equation (if the origin is so chosen that $\mathrm{U} = \beta y$) takes the form

$$[\mathrm{Q} + \nabla^2 - i(\mathrm{P} + \mathrm{B}z)]\,\nabla^2\Psi = 0, \qquad \ldots \ldots \ldots \ldots \quad (3$$

where ∇^2 now stands for the operator $[d^2/dz^2 - 1]$, and Ψ is a function of z which, by (5) of Section III, must satisfy the boundary conditions

$$\Psi = \frac{d\Psi}{dz} = 0. \qquad (4)$$

If we write

$$\Psi = \Psi_R + i\Psi_I, \qquad (5)$$

where Ψ_R and Ψ_I are real functions of z, a real form of solution corresponding to (1) is

$$\psi = e^{-qt}\{\Psi_R \sin(kx + pt) + \Psi_I \cos(kx + pt)\}. \qquad (6)$$

Substituting from (5) in (3), we see that Ψ_R and Ψ_I are subject to the relations

$$\left.\begin{aligned} [Q + \nabla^2]\,\nabla^2\Psi_R + (P + Bz)\,\nabla^2\Psi_I = 0, \\ [Q + \nabla^2]\,\nabla^2\Psi_I - (P + Bz)\,\nabla^2\Psi_R = 0, \end{aligned}\right\} \qquad (7)$$

and, by (4), to the boundary conditions

$$\Psi_R = \frac{d\Psi_R}{dz} = \Psi_I = \frac{d\Psi_I}{dz} = 0.$$

These are the required equations.

Solution in the Special Case U = 0.

2. It is easy to show, from (7), that P must be zero when U (and hence B) = 0. Multiplying the first and second of (7) by Ψ_I and Ψ_R respectively, subtracting and integrating over the whole breadth of the fluid field, we obtain, in virtue of the boundary conditions, the relation

$$P\int\{\Psi'^2_R + \Psi'^2_I + \Psi_R^2 + \Psi_I^2\}\,dz = 0, \qquad (8)$$

where dashes denote differentiations with respect to z. Since the integrand in (8) is composed of squares of real quantities, both Ψ_R and Ψ_I must vanish at all points in the field, unless P = 0.

3. It follows from (7) that, in the special case now considered, Ψ_R and Ψ_I are independent solutions of the equation

$$[Q + \nabla^2]\,\nabla^2\Psi = 0 \qquad (9)$$

which satisfy the stated boundary conditions. If we take the origin at the centre of the field, so that the boundaries are represented by

$$z = \pm kb = \pm\gamma, \qquad (10)$$

it is evident that solutions of (9) may be either purely odd or purely even in z. Further, if we write

$$\alpha^2 = Q - 1, \quad \ldots \ldots \quad (11)$$

the even solution takes the form

$$\Psi = A \cosh z + B \cos \alpha z, \quad \ldots \ldots \quad (12)$$

where

$$\frac{\cot \alpha\gamma}{\alpha\gamma} = -\frac{\coth \gamma}{\gamma} \quad \ldots \ldots \quad (13)$$

and

$$A \cosh \gamma + B \cos \alpha\gamma = 0, \quad \ldots \ldots \quad (14)$$

and the odd solution takes the form

$$\Psi = C \sinh z + D \sin \alpha z, \quad \ldots \ldots \quad (15)$$

where

$$\frac{\tan \alpha\gamma}{\alpha\gamma} = \frac{\tanh \gamma}{\gamma} \quad \ldots \ldots \quad (16)$$

and

$$C \sinh \gamma + D \sin \alpha\gamma = 0. \quad \ldots \ldots \quad (17)$$

Solutions of (13) and (16), regarded as equations in $\alpha\gamma$, are easily obtained by "trial and error" when γ is assumed. The roots of (13) are separated by roots of (16), the lowest value of $\alpha\gamma$ coming from equation (13). As γ increases (*i.e.*, as the wave-length of the disturbance becomes less in relation to the breadth of the fluid field), all the roots steadily decrease. The variation of the graver roots is indicated in fig. 4.

Solution when U *is Finite. Limitations on the Magnitude of* P.

4. When U (and therefore B) is finite, it is no longer possible to assert that P must be zero. In place of equation (8), we now obtain from (7) the condition

$$\int (P + Bz) \{\Psi'^2_R + \Psi'^2_I + \Psi_R^2 + \Psi_I^2\}\, dz = 0, \quad \ldots \ldots \quad (18)$$

where dashes denote, as before, differentiations with respect to z. The conclusion to be drawn from this modified result is that $(P + Bz)$—that is, $(p + k\beta y)$—must change sign within the fluid field: this has been remarked by ORR.*

The ratio $-p/k$ can be seen, from (6), to be the velocity with which the disturbance is propagated along the fluid in the direction of the x-axis; and since the origin has been so chosen that the velocity U of the steady stream is given by

$$U = \beta y,$$

our restriction on the value of p may be interpreted as requiring that *this velocity of*

* *Loc. cit.*, p. 99.

propagation shall be equal to the velocity of the steady stream at some point within the breadth of the fluid field.

A Simplifying Assumption.

5. The investigation of § 2 showed that P is zero,—*i.e.*, that the time factor of the disturbance is purely real,—when the boundaries are at rest. If we take the origin at the centre of the field (so that the steady shearing motion involves equal and opposite

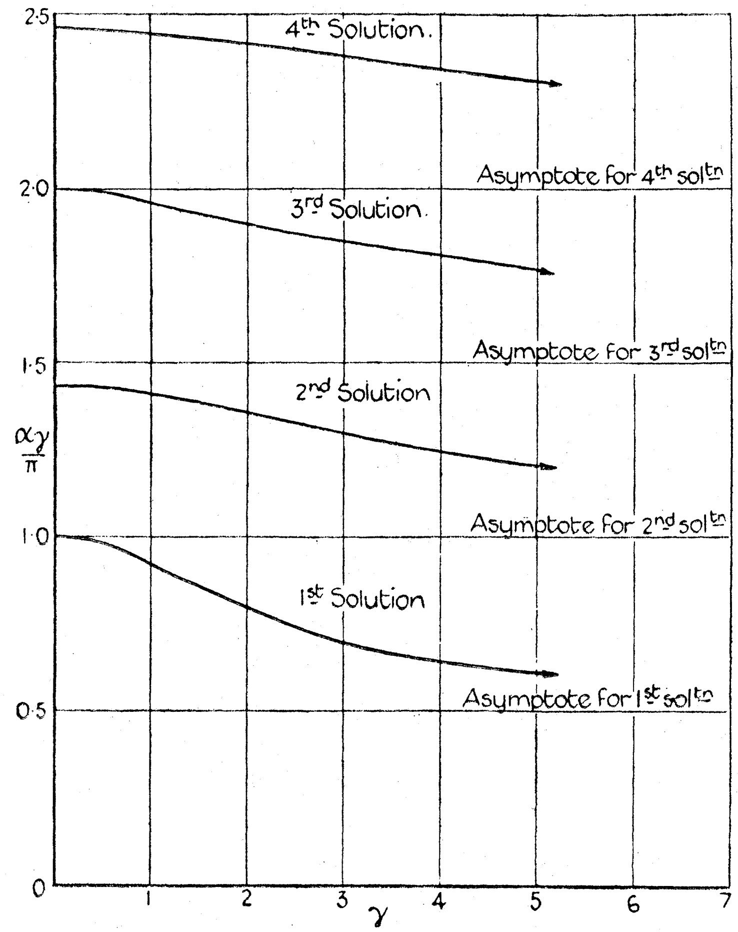

FIG. 4.—Curve relating $\alpha\gamma$ with γ where $(\alpha^2 + 1)\gamma^2 = Q\gamma^2 = qb^2/\nu$.

velocities at the two boundaries), it is thus reasonable to expect that the normal disturbances will be characterised by a purely real time factor over part at least of the range $(0 < B < \infty)$. The numerical calculations were started on this assumption; that is to say, the boundaries were taken to be given by

$$z = \pm kb = \pm \gamma, \quad \ldots \ldots \ldots \ldots \quad (10)\ bis$$

and it was assumed that $P = 0$. This assumption underlies all the results which follow.

Nature of the Normal Disturbances when P = 0.

6. On this assumption, equations (7) become

$$\left.\begin{aligned}[Q + \nabla^2]\,\nabla^2\Psi_R + Bz\nabla^2\Psi_I &= 0,\\ [Q + \nabla^2]\,\nabla^2\Psi_I - Bz\nabla^2\Psi_R &= 0,\end{aligned}\right\} \quad \ldots \quad (19)$$

where ∇^2, as before, denotes the operator $[d^2/dz^2 - 1]$. The boundary conditions are

$$\Psi_R = \Psi'_R = \Psi_I = \Psi'_I = 0, \quad \text{when} \quad z = \pm\gamma. \quad \ldots \quad (20)$$

The form of these equations, considered in conjunction with (10), indicates that we may assume the normal disturbance to be of a type in which Ψ_R is an even function and Ψ_I an odd function of z. For the even part of Ψ_R, together with the odd part of Ψ_I, constitute a solution in no way dependent on that which is constituted by the odd part of Ψ_R and the even part of Ψ_I; and further, the second of these two solutions, taken separately, can be seen from (6) to represent a disturbance which differs from the first solution only in respect of its phase regarded as a function of x.

Derivation of the "Characteristic Equation."

7. Accordingly we assume for Ψ_R and Ψ_I the infinite power series represented by

$$\left.\begin{aligned}\Psi_R &= \Sigma\left[a_{2n}\frac{z^{2n}}{2n\,!}\right],\\ \Psi_I &= \Sigma\left[b_{2n+1}\frac{z^{2n+1}}{2n+1\,!}\right].\end{aligned}\right\} \quad \ldots \quad (21)$$

Substituting these expressions in (19), we obtain the relations

$$\left.\begin{aligned}a_{2n+4} &= -(Q-2)\,a_{2n+2} + (Q-1)\,a_{2n} - 2nB\,(b_{2n+1} - b_{2n-1}),\\ b_{2n+5} &= -(Q-2)\,b_{2n+3} + (Q-1)\,b_{2n+1} + (2n+1)\,B\,(a_{2n+2} - a_{2n}),\end{aligned}\right\} \quad \ldots \quad (22)$$

when $n \geqslant 0$. From these the coefficients a_{2n}, b_{2n+1} can be determined when a_0, b_1, a_2, b_3 have specified values. Thus we can obtain four independent solutions of (19), each of arbitrary magnitude.

We denote these four solutions by Ψ_1, Ψ_2, Ψ_3, Ψ_4, and derive them from the following assumptions :—

$$\left.\begin{aligned}&\text{for } \Psi_1 : \; a_0 = 1, && a_2 = b_1 = b_3 = 0,\\ &\text{for } \Psi_2 : \; a_0 = a_2 = 1, && b_1 = b_3 = 0,\\ &\text{for } \Psi_3 : \; b_1 = 1, && a_0 = a_2 = b_3 = 0,\\ &\text{for } \Psi_4 : \; b_1 = b_3 = 1, && a_0 = a_2 = 0.\end{aligned}\right\} \quad \ldots \quad (23)$$

The most general solution may now be written in the form

$$\Psi = A\Psi_1 + B\Psi_2 + C\Psi_3 + D\Psi_4, \quad \ldots \ldots \ldots \quad (24)$$

where A, B, C, D are arbitrary constants; and the boundary conditions yield four relations between A, B, C, D, since the real and imaginary parts of (24) must satisfy these conditions separately. If we denote, *e.g.*, the real part of Ψ_1 by ${}_1\Psi_R$ and the imaginary part of Ψ_1 by ${}_1\Psi_I$, and eliminate A, B, C, D from these four relations, we obtain the "characteristic equation"

$$\Delta = \begin{vmatrix} {}_1\Psi_R, & {}_2\Psi_R, & {}_3\Psi_R, & {}_4\Psi_R \\ {}_1\Psi'_R, & {}_2\Psi'_R, & {}_3\Psi'_R, & {}_4\Psi'_R \\ {}_1\Psi_I, & {}_2\Psi_I, & {}_3\Psi_I, & {}_4\Psi_I \\ {}_1\Psi'_I, & {}_2\Psi'_I, & {}_3\Psi'_I, & {}_4\Psi'_I \end{vmatrix} = 0, \quad \text{when } z = \gamma. \quad (25)$$

Now it can be seen from (22) and (23) that

$$\left.\begin{aligned} \Psi_2 &= \cosh z, \\ \Psi_4 &= \sinh z. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (26)$$

Accordingly, equation (25) may be simplified in form; it becomes

$$-\Delta = \cosh^2 z \begin{vmatrix} {}_1\Psi'_R - {}_1\Psi_R \tanh z, & {}_3\Psi'_R - {}_3\Psi_R \tanh z \\ {}_1\Psi_I - {}_1\Psi'_I \tanh z, & {}_3\Psi_I - {}_3\Psi'_I \tanh z \end{vmatrix} = 0, \quad \text{when } z = \gamma. \quad (27)$$

Solution of the Characteristic Equation.

8. Equation (27) has been solved by a process of trial and error. A definite value of γ is assumed. Next, for some selected value of B, the coefficients in the series for Ψ_1 and Ψ_3 are calculated by means of the relations (22), and the determinant in (27) is evaluated for a series of values of Q.* Then, on plotting a curve to show the variation of Δ with Q (*e.g.*, fig. 5), we find that Δ oscillates about the zero line, so that "critical" values of Q can be determined for which Δ vanishes. Three of these critical values are given by fig. 5, which has been drawn for $B = 1$, $\gamma = 2$.†

Repeating this procedure for other selected values of B, we obtain material for the construction of a curve relating (for the assumed value of γ) B with the critical values

* In some instances it is convenient to work with a single selected value of Q and a series of values of B; but the procedure is the same.

† A preliminary estimate of the values of Q for which Δ will vanish can be formed, for small values of B, from inspection of fig. 4.

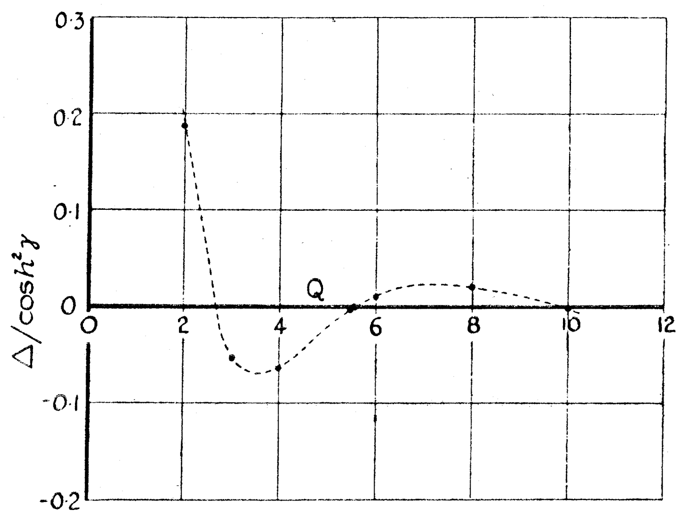

Fig. 5.—Curve relating Q with Δ, for $\gamma = 2$, B = 1.

of Q. Fig. 6 illustrates this relation (in the case of the four lowest critical values of Q) when $\gamma = 2$: results for other values of γ will be given later.

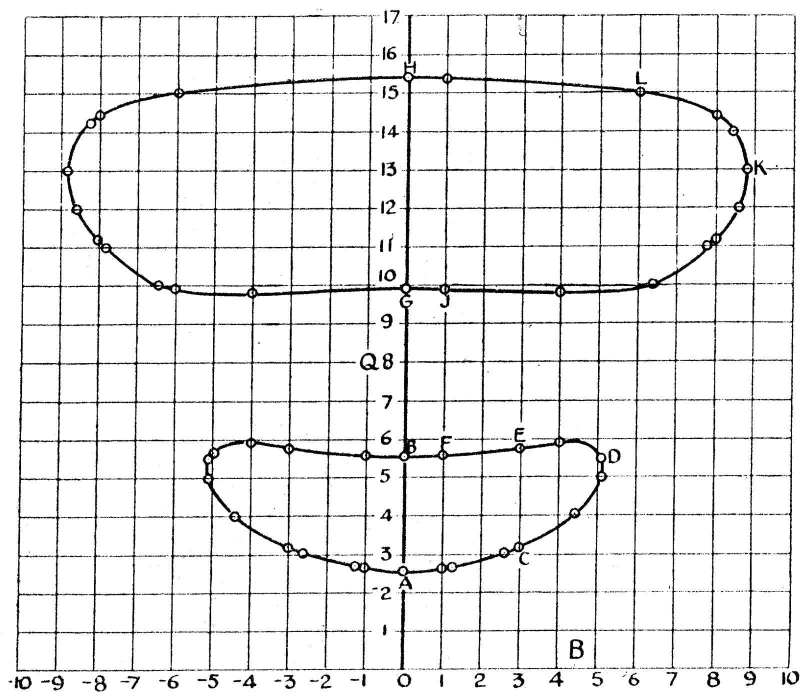

Fig. 6.—Relation between Q and B for $\gamma = 2$.

Discussion of Results.

9. The outstanding feature of the Q — B relation, as exemplified by fig. 6, is the looped form of the curves. This has no counterpart in any ordinary problem of vibrations that has come within our notice, but it can be reproduced in simple systems of an artificial

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nature. The continuous transition between points A and B on the lower loop of fig. 6, by way of the points C, D, E, F, would seem to imply a continuous transition in the type of the normal disturbance. The type corresponding with A is that defined by equations (12) to (14) of § 3, in which Ψ has the same sign in both halves of the field: the type corresponding with point B is that defined by equations (15) to (17), in which Ψ changes sign at the centre of the field.

When these results were first obtained, it seemed difficult to imagine any process of continuous transition from the first type of disturbance to the second, and attention was accordingly focussed on the nature of the normal disturbances implied by our solutions. We pass now to a description of the processes by which these disturbances were investigated.

Nature of the Normal Disturbances. Determination of Contours.

10. Equation (25) is derived from the boundary conditions (20) by elimination of the arbitrary constants A, B, C, D,—*i.e.*, of the three ratios B/A, C/A, D/A. Any three of the boundary conditions will serve to determine these three ratios, and the fact that (25) has been satisfied implies that the fourth boundary condition will be consistent with the values so found. Thus it was a straightforward (although lengthy) matter to determine the ratios appropriate to any point in the curves of fig. 6, and thence to determine the appropriate forms of Ψ_R and Ψ_I.

It was then possible to calculate the form of any contour of constant ψ according to equation (6). When $t = 0$, this expression for ψ may be thrown into the form

$$\psi = \sqrt{\Psi_R^2 + \Psi_I^2} \sin \{\gamma (x/b) + \tan^{-1} (\Psi_I/\Psi_R)\},$$

and hence, knowing Ψ_R and Ψ_I for any value $\bar{z}$ $(= \gamma\, \bar{y}/b)$, and given the value of ψ for the contour which is to be plotted, we can find the values of $\gamma\, \bar{x}/b$ for the points at which this contour cuts the line $z = \pm z$.

Figs. 7 to 17 show the results of calculations conducted on the foregoing lines in relation to the lettered points of fig. 6. These contour diagrams clear up completely the difficulty mentioned in § 9 as associated with the idea of continuous transition from modes in which Ψ is an even function to modes in which Ψ is an odd function of z; and the diagrams included in the lower parts of the figures, which give the distribution of Ψ_R and Ψ_I across the section, show how the relative magnitudes of the even and odd functions vary as we pass round the loops of fig. 6. In the first loop, starting at A with a one-signed even function (fig. 7), we find that, on the imposition of a gradually increasing B, an odd component having a single sign in each half of the field is introduced (fig. 8). This component gradually increases in relative magnitude, whilst the even component gradually diminishes; and after a point D (fig. 6) has been passed, the same process continues with *decreasing* B, until at the point B (B = 0, fig. 12) the function has become purely odd, and of one sign in each half of the field.

(A) Contours of Constant ψ.—The numbers against the contours give proportion of ψ_{max}.

FIG. 7.—Point A.

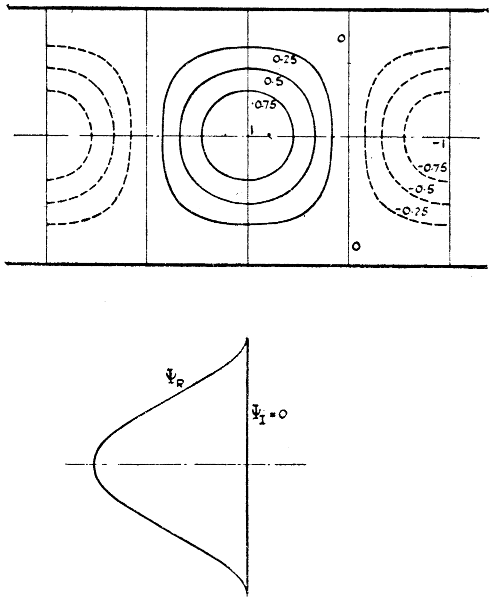


FIG. 8.—Point C.

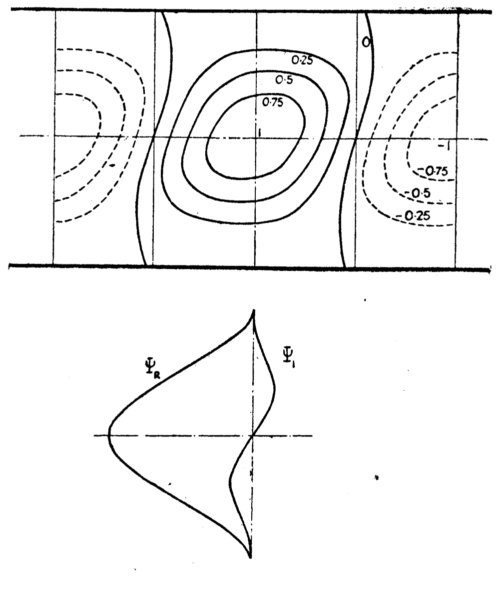
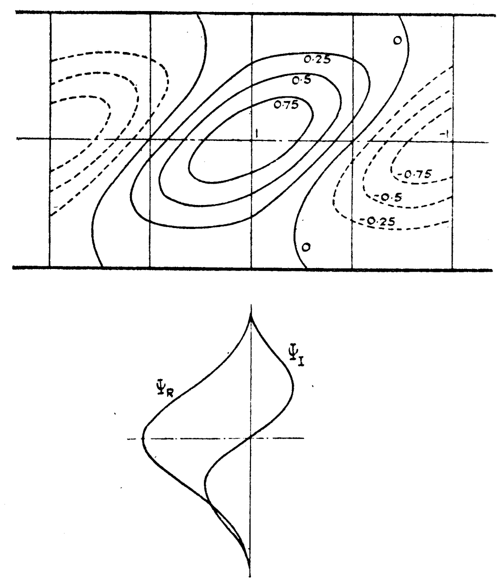


FIG. 9.—Point D.



(B) Distribution of Ψ_R and Ψ_I across the Field. (N.B.—Absolute magnitudes are indeterminate.)

(A) Contours of Constant ψ.—The numbers against the contours give proportion of ψ_{max}.

FIG. 10.—Point E.

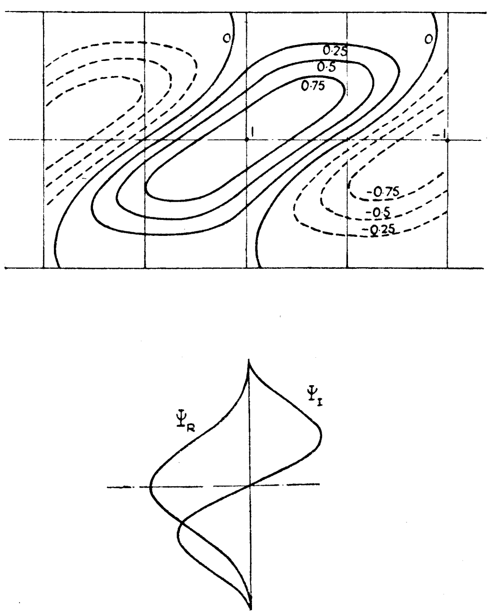


FIG. 11.—Point F.

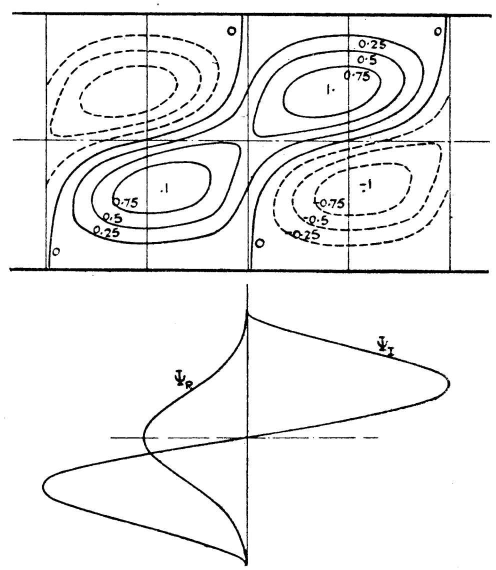
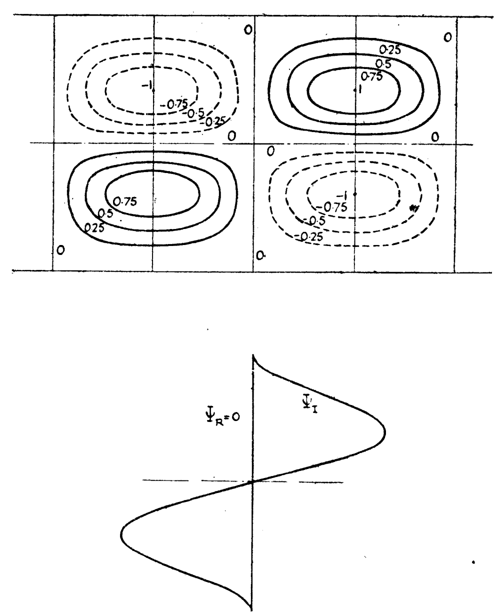


FIG. 12.—Point B.



(B) Distribution of Ψ_R and Ψ_I across the Field. (N.B.—Absolute magnitudes are indeterminate.)

The same process is observable in the second loop of fig. 6, with slightly greater complication. Starting at the point G (B = 0, fig. 13) with an even function which changes sign on either side of the centre line, we find that as B increases an odd component —having a single sign in either half of the field—is introduced (fig. 14). This component increases in relative magnitude, and gradually changes into an odd function which has both signs in each half of the field (fig. 15). After the point K has been passed, the same process is maintained with diminishing B (fig. 16), until at the point H (B = 0, fig. 17) the whole function is purely odd, with both signs in each half of the field.

Additional Results.

11. Calculations on the lines described in §§ 7 and 8 have also been made to determine the forms of the "Q — B loops" for $\gamma = 1$ and for $\gamma = 3$. The results, with those for $\gamma = 2$, have been plotted in fig. 1 on a slightly different basis, $Q\gamma^2$ (or $qd^2/4\nu$) being related with $4B\gamma^2$ (or $\beta d^2/\nu$), which is REYNOLDS' number for our problem (§ 1).

General Remarks on Method.

12. In practice it was found that successive terms in the a and b series of (21) were of alternate sign, diminishing (after a time) in magnitude. A fair approximation to the value of Δ could be obtained by retaining coefficients in (21), for the first loop as far as $n = 12$, and for the second loop as far as $n = 16$. The coefficients are required (in some cases) correct to seven figures; expressions for Ψ_R, etc., can then be obtained correctly to five figures, and the resulting estimate of Δ is correct to two or three figures.* As Q increases for a given B (or as B increases for a given Q), the coefficients increase in magnitude, and it becomes necessary to take more terms into account in order to obtain the same degree of accuracy in evaluating Ψ_R etc. Since Δ is found to oscillate with diminishing amplitude as Q increases, a higher degree of accuracy is required in determining Ψ_R etc., in order to obtain satisfactory approximation in Δ.

For this reason the method becomes impracticable when Q and B are large, and the range of our exploration is accordingly very limited. In fact, as fig. 1 makes clear, our curves are confined in the main within a range of REYNOLDS' number which is less than ORR's figure (177) for the lowest value at which instability can possibly occur. Thus we could have foretold that Q would be positive throughout the greater part of our diagrams, and the actual determination of the loops has accordingly thrown little new light on the question of stability or instability.

It is clear that, when R has values such that instability can be contemplated, we must discard the assumption made throughout this investigation,—that P is zero when the

* These details relate to the calculations made for $\gamma = 2$.

(A) Contours of Constant ψ.—The numbers against the contours give proportion of ψ_{max}.

FIG. 13.—Point G.

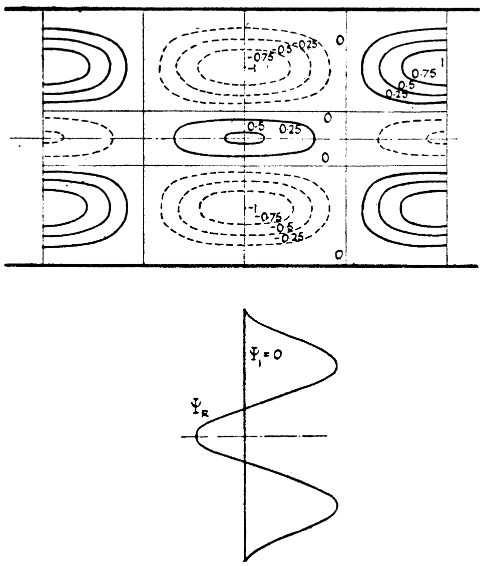


FIG. 14.—Point J.

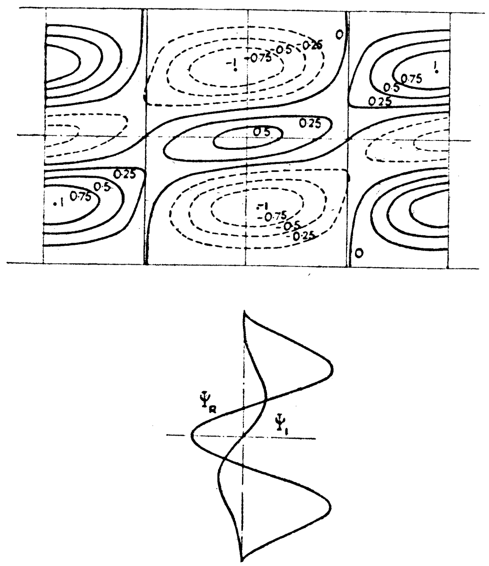
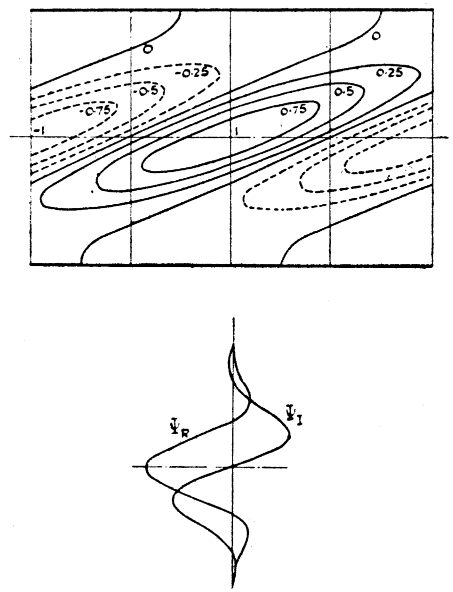


FIG. 15.—Point K.



(B) Distribution of Ψ_R and Ψ_I across the Field. (N.B.—Absolute magnitudes are indeterminate.)

(A) Contours of Constant ψ.—The numbers against the contours give proportion of ψ_{max}.

FIG. 16.—Point L.

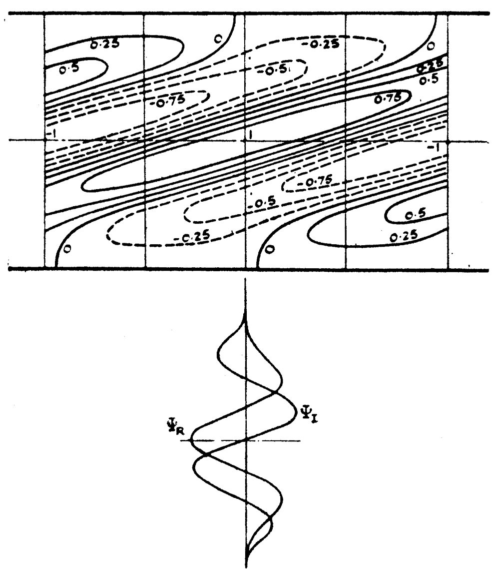
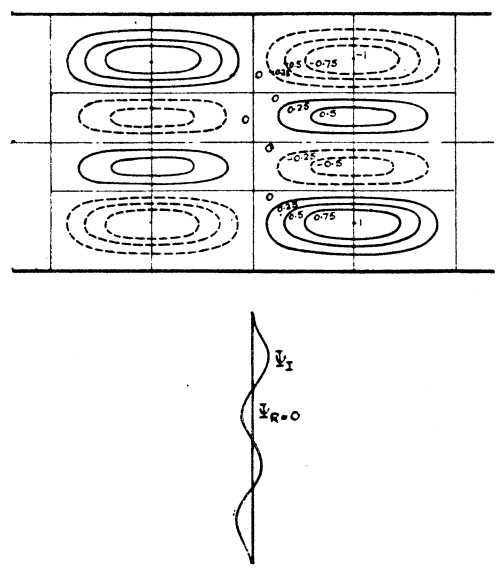


FIG. 17.—Point H.



(B) Distribution of Ψ_R and Ψ_I across the Field. (N.B.—Absolute magnitudes are indeterminate.)

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origin is at the centre of the stream. Now if P is finite but unknown, we have an additional parameter to be varied in deriving solutions by our process of trial and error. It will be appreciated that this additional complication would increase the labour of calculation to an intolerable extent, having regard to the fact that, as just stated, we are already near the practical limits of our method. It is thus essential, if further progress is to be made, to develop alternative lines of attack. A subsequent paper will describe our attempts in this direction.

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